Commutative Algebra
Chapter 5: Integral Dependence and Valuations

Sichuan University, Fall 2020

Integral Dependence

Reminder

Let k be a field.

- An element x of some field extension of k is said to be *algebraic over* k if it is the root of some polynomial equations with coefficients in k, i.e.,

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in k, \ a_n \neq 0.$$

- An *algebraic extension* of k is a field extension L of k in which every element is algebraic over k.
- We say that k is *algebraically closed* when the all the roots of every polynomial equations with coefficients in k are contained in k.
- Every field admits an algebraically closed extension.

Integral Dependence

Definition

Let $A \subseteq B$ be rings. We say that $x \in B$ is *integral over* A if it solution of a *monic* polynomial equation with coefficients in A, i.e., an equation of the form,

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in A.$$

Remark

Every $x \in A$ is integral over A.

Example

Let $A = \mathbb{Z}$ and $B = \mathbb{Q}$. Then $x \in \mathbb{Q}$ is integral over $\mathbb{Z}$ if and only if $x \in \mathbb{Z}$.

Integral Dependence

Proposition (Proposition 5.1)

Let $A \subseteq B$ be rings and $x \in B$. TFAE:

(i) *x is integral over A.*

(ii) *$A[x]$ is a finitely generated A-module.*

(iii) *$A[x]$ is contained in a subring C of B such that C is a finitely generated A-module.*

(iv) *There is a faithful $A[x]$-module M which is finitely generated as an A-module.*

Reminder (Faithful module; see Chapter 2)

A module M over A is *faithfull* when its annihilator is zero, i.e., if $a \in A$, then

$$ax = 0 \quad \forall x \in M \implies a = 0.$$

Integral Dependence

Reminder (Proposition 2.4; see Gaillard)

Let M be a finitely generated A-module and $\mathfrak{a}$ and ideal of A. Let $\phi : M \to M$ be an A-module endomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form,

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \qquad a_i \in \mathfrak{a}.$$

In particular, for $\mathfrak{a} = A$ we get:

Corollary

Let M be a finitely generated A-module. Then any A-module endomorphism $\phi : M \to M$ satisfies an equation of the form,

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \qquad a_i \in A.$$

Integral Dependence

Reminder (Proposition 2.16)

Let $A \subseteq B$ be rings. If M is a finitely generated B-module and B is finitely generated as an A-module, then M is finitely generated as an A-module.

Corollary (Corollary 5.2)

Let $x_1, \ldots, x_n$ be elements of B that are integral over A. Then the ring $A[x_1, \ldots, x_n]$ is a finitely generated A-module.

Corollary (Corollary 5.3)

The set of all elements of B that are integral over A forms a sub-ring of B containing A.

Definition (Integral closure)

- The sub-ring of elements of B that are integral over A is called the *integral closure of* A in B and is denoted $B * A$ (Gaillard's notation).
- We say that A is *integrally closed* if $B * A = A$.
- We say that B is *integral over* A if $B * A = B$.

Integral Dependence

Reminder (Finite and finite-type algebras; see Chapter 2)

Let B be an A-algebra.

- We say that the algebra B is *finite* if it is finitely generated as an A-module.
- We say that the algebra B has *finite type* if $B = A[x_1, \ldots, x_n]$ for some $x_i \in B$.

Remark

It follows from Corollary 5.2 that if an A-algebra B has finite type and is integral over A, then B is a finite A-algebra.

Corollary (Corollary 5.4)

Let $A \subseteq B \subseteq C$ be rings such that B is integral over A and C is integral over B. Then C is integral over A.

Corollary (Corollary 5.5)

Let $A \subseteq B$ be rings. Then $B * A$ is integrally closed in B.

Proposition (Proposition 5.6)

Let $A \subseteq B$ be rings such that B is integral over A.

(i) *If $\mathfrak{b}$ is an ideal of B and $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b} \cap A$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.*

(ii) *Let S be a multiplicatively closed subset of A. Then $S^{-1}B$ is integral over $S^{-1}A$.*

The Going-Up Theorem

Reminder (Integral domains; see Chapter 1)

A ring A is called an *integral domain* if

$$xy = 0 \Longrightarrow x = 0 \text{ or } y = 0.$$

Proposition (Proposition 5.7)

Let $A \subset B$ be integral domains such that B is integral over A. Then

$$B \text{ is a field} \iff A \text{ is a field}.$$

The Going-Up Theorem

Reminder (Prime and maximal ideals; see Chapter 1)

Let $\mathfrak{p}$ be an ideal of a ring A. Then

$$\begin{aligned} \mathfrak{p} \text{ is prime} &\iff A/\mathfrak{p} \text{ is an integral domain}, \\ \mathfrak{p} \text{ is maximal} &\iff A/\mathfrak{p} \text{ is a field}, \end{aligned}$$

The Going-Up Theorem

Corollary (Corollary 5.8)

Let $A \subseteq B$ be rings such that B is integral over A. Let $\mathfrak{q}$ be a prime ideal of B and set $\mathfrak{p} = \mathfrak{q}^c = \mathfrak{q} \cap A$. Then

$$\mathfrak{q} \text{ is maximal} \iff \mathfrak{p} \text{ is maximal.}$$

Remark (Contractions of ideals; see Chapter 1)

Let $A \subset B$ be rings. The inclusion of A into B is a ring homomorphism. Thus, if $\mathfrak{b}$ is an ideal of B, then its contraction in A is $\mathfrak{b}^c = \mathfrak{b} \cap A$.

The Going-Up Theorem

Reminder (Rings of fractions; Corollary 3.4 and Proposition 3.11)

Let S be a multiplicatively closed subset of a ring A.

- If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A, then $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})$.
- There is a one-to-correspondence $(\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p})$ between the prime ideals of $S^{-1}A$ and the prime ideals of A that don't meet S.
- In particular, if $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals of A that don't meet S, then $S^{-1}\mathfrak{p} = S^{-1}\mathfrak{p}' \Rightarrow \mathfrak{p} = \mathfrak{p}'$.
- If $S = A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A, then $S^{-1}\mathfrak{p}$ is the maximal ideal of the local ring $A_{\mathfrak{p}} = S^{-1}A$.

Corollary (Corollary 5.9)

Let $A \subseteq B$ be rings such that B is integral over A. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be prime ideals of B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$. Then $\mathfrak{q} = \mathfrak{q}'$.

The Going-Up Theorem

Theorem (Theorem 5.10)

Let $A \subseteq B$ be rings such that B is integral over A. Then, for any prime ideal $\mathfrak{p}$ of A, there is a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.

Theorem (Going-Up Theorem; Theorem 5.11)

Let $A \subseteq B$ be rings such that B is integral over A. Suppose we are given the following:

- *A chain $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ of prime ideals of A.*
- *A chain $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m$ of prime ideals of B with $m < n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, m$.*

Then the latter chain extends to a chain $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_n$ of ideals of B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, n$.

Proposition (Proposition 5.12)

Let $A \subseteq B$ be rings and S a multiplicatively closed subset of A. Then $S^{-1}(BA)$ is the integral closure of $S^{-1}A$ in $S^{-1}B$, i.e.,

$$\left(S^{-1}B\right) * \left(S^{-1}A\right) = S^{-1}(B * A).$$

Integrally Closed Domains. Going-Down Theorem

Reminder (Fraction Field; slides on Chapter 3)

If A is an integral domain, its *field of fraction*, denoted $\mathrm{Frac}(A)$, is $S^{-1}A$ with $S = A \setminus 0$.

Definition

A say that an integral domain A is *integrally closed* when it is integrally closed in its fraction ring $\mathrm{Frac}(A)$.

Example

The ring $A = \mathbb{Z}$ is an integral domain with fraction field $\mathbb{Q}$ and it is integrally closed in $\mathbb{Q}$ (see slide 2). Thus, $\mathbb{Z}$ is an integrally closed integral domain.

More generally, any principal domain with the unique factorization property is integrally closed. In particular, we have:

Example

Any polynomial ring $A = k[x_1, \dots, x_n]$ over a field k is integrally closed.

Reminder (Surjectivity is a local property; Proposition 3.9)

Let $\phi : M \rightarrow N$ be an A-module homomorphism between A-modules. Then TFAE:

1. ϕ is surjective.
2. $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is surjective for every prime ideal $\mathfrak{p}$ of A.
3. $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m}$ of A.

Integrally Closed Domains. Going-Down Theorem

Integral closure is a local property:

Proposition (Proposition 5.13)

Let A be an integral domain. Then TFAE:

(i) *A is integrally closed.*

(ii) *$A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$.*

(iii) *$A_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m}$.*

Remark

Let A be an integral domain and $\mathfrak{p}$ a prime ideal of A. Then:

- The local ring $A_{\mathfrak{p}}$ is an integral domain.
- The natural ring homomorphism $A \to A_{\mathfrak{p}}$ is an injection.
- It can be shown that the fractions fields of A and $A_{\mathfrak{p}}$ agree. (This follows from the functorial properties of fraction rings; exercise!)

Definition

Let $A \subseteq B$ be rings and $\mathfrak{a}$ an ideal of $\mathfrak{a}$.

- An element $x \in B$ is said to be integral over $\mathfrak{a}$ if it is solution of monic equation with coefficients in $\mathfrak{a}$, i.e.,

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in \mathfrak{a}.$$

- The set of all such elements is called the *integral closure of* $\mathfrak{a}$ *in* B and is denoted $B * \mathfrak{a}$ (Gaillard's notation).

Remark (Contractions of ideals; see Chapter 1)

Let $A \subset B$ be rings. The inclusion of A into B is a ring homomorphism. Therefore:

- If $\mathfrak{a}$ is an ideal of A, then its extension in B is $\mathfrak{a}^e = B\mathfrak{a}$, i.e., it consists all finite sums $\sum b_i a_i$ with $b_i \in B$ and $a_i \in \mathfrak{a}$.
- If $\mathfrak{b}$ is an ideal of B, then its contraction in A is $\mathfrak{b}^c = \mathfrak{b} \cap A$.

Lemma (Lemma 5.14)

Let $A \subseteq B$ be rings and $\mathfrak{a}$ an ideal of $\mathfrak{a}$. Then the integral closure of $\mathfrak{a}$ in B is the radical of its extension in B. That is,

$$B * \mathfrak{a} = r(B\mathfrak{a}).$$

In particular, $B * \mathfrak{a}$ is an ideal of B.

Proposition (Proposition 5.15)

Let $A \subseteq B$ be integral domains such that A is integrally closed. Let $x \in B$ be integral over an ideal $\mathfrak{a}$ of A.

1. *x is algebraic over the fraction field $K = \mathrm{Frac}(A)$.*
2. *Let $\mu(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be the minimal polynomial of x over K. Then all the coefficients $a_1, \ldots, a_n$ lie in $r(\mathfrak{a})$.*

Integrally Closed Domains. Going-Down Theorem

Reminder (Contracted ideals; see Proposition 1.17(iii))

Let $f : A \to B$ be a ring homomorphism.

- An ideal $\mathfrak{a}$ of A is a the contraction of an ideal of B if and only if $\mathfrak{a}^{ec} = \mathfrak{a}$.
- In particular, if $A \subseteq B$ and f is the inclusion map, then the above condition amounts to

$$(B\mathfrak{a}) \cap A = \mathfrak{a}.$$

Reminder (Prime ideals of localisations; see Chapter 3)

Let A be a ring and $\mathfrak{p}$ a prime ideal. Then we have a one-to-one correspondence between prime ideals of $A_{\mathfrak{p}}$ and prime ideals of A contained in $\mathfrak{p}$.

Integrally Closed Domains. Going-Down Theorem

Theorem (Going-Down Theorem; Theorem 5.16)

Let $A \subseteq B$ be integral domains such that A is integrally closed and B is integral over A, i.e., $K * A = A$ and $B * A = B$, where $K = \mathrm{Frac}(A)$. Assume we are given the following:

- *A chain $\mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_n$ of prime ideals of A.*
- *A chain $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_m$ of prime ideals of B with $m < n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, m$.*

Then the latter chain extends to a chain $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_n$ of ideals of B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i = 1, \ldots, n$.

Valuation Rings

Definition

We say that a ring B is a *valuation ring* of a field K if K contains B as a sub-ring and

$$x \in K \setminus 0 \implies x \in B \text{ or } x^{-1} \in B.$$

Remarks

1. Any sub-ring of a field is automatically an integral domain, and hence any valuation ring is an integral domain.
2. If B is a valuation ring for a field K, then K must be the field of fractions of B. (This follows from the functorial properties of fraction fields; exercise!).

Valuation Rings

Reminder (Characterization of local rings; see Proposition 1.6(ii))

Let A be a ring and $\mathfrak{m} \neq A$ an ideal of A such that every $x \in A \setminus \mathfrak{m}$ is a unit in A. Then A is a local ring and $\mathfrak{m}$ is its maximal ideal.

Proposition (Proposition 5.18)

Let B a valuation ring in a field K.

(i) *B is a local ring.*

(ii) *Any sub-ring of B is a valuation ring of K.*

(iii) *B is integrally closed in K.*

Facts

Let K be a field and Ω an algebraically closed field.

- Define Σ to be the set of pairs (A, f), where A is a sub-ring of K and $f : A \to \Omega$ is a ring homomorphism.
- Σ is a partially ordered set:

$$(A, f) \leq (A', f') \iff A \subseteq A' \text{ and } f'_{|A} = f.$$

- By Zorn's lemma Σ admits a maximal element.

Valuation Rings

Theorem (Theorem 5.21; see Atiyah-MacDonald)

If (B, g) is a maximal element of Σ, then the ring B is a valuation ring of K.

Corollary (Corollary 5.22)

Let A be a sub-ring of a field K. Then the integral closure $K * A$ is the intersections of all the valuation rings of K that contain A.

Valuation Rings

Proposition (Proposition 5.23)

Let $A \subseteq B$ be integral domains such that B is finitely generated over A. Let $v \in B \setminus 0$. Then there is $u \in A \setminus 0$ with the following property: any homomorphism f of A into an algebraically closed field Ω such that $f(u) \neq 0$ extends to a homomorphism $g : B \to \Omega$ such that $g(v) \neq 0$.

Corollary (Corollary 5.24)

Let k be a field and B a finitely generated k-algebra. If B is a field, then it is a finite algebraic extension of k.

Corollary (Weak Nullstellensatz; Corollary 7.10)

Let k be a field, A a finitely generated k-algebra, and $\mathfrak{m}$ a maximal ideal of A. Then the field $A/\mathfrak{m}$ is a finite algebraic extension of k. In particular, if k is algebraically closed, then $A/\mathfrak{m} \simeq k$.