Commutative Algebra
Chapter 4: Primary Decomposition

Sichuan University, Fall 2020

Primary Decomposition

Reminder (Prime ideals; see Chapter 1)

- An ideal $\mathfrak{p}$ in a ring A is *prime* if $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.
- Equivalently, $\mathfrak{p}$ is prime if and only if $A/\mathfrak{p}$ is an integral domain.
- Every maximal ideal is prime.

Primary Decomposition

Definition (Primary Ideals)

An ideal $\mathfrak{q}$ in A is *primary* if $\mathfrak{q} \neq A$ and

$$xy \in \mathfrak{q} \Longrightarrow x \in \mathfrak{q} \text{ or } y^n \in \mathfrak{q} \text{ for some } n \geq 1.$$

Remarks

1. If $\mathfrak{q} \neq A$, then
$$\mathfrak{q} \text{ primary} \Longleftrightarrow \text{every zero-divisor in } A/\mathfrak{q} \text{ is nilpotent.}$$
2. Every prime ideal is primary.
3. If $f : A \rightarrow B$ is a ring homomorphism and $\mathfrak{q}$ is a primary ideal in B, then its contraction $\mathfrak{q}^c = f^{-1}(\mathfrak{q})$ is a primary ideal in A.

Primary Decomposition

Reminder (Radicals of ideals; see Chapter 1)

- If $\mathfrak{a}$ is an ideal in A, then its *radical* is

$$r(\mathfrak{a}) = \{x \in A;\ x^n \in \mathfrak{a} \text{ for some } n \geq 1\}.$$

- $r(\mathfrak{a})$ is the intersection of all the prime ideals that contain $\mathfrak{a}$. In particular, this is an ideal containing $\mathfrak{a}$.
- $r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.
- If $\mathfrak{p}$ is a prime ideal, then $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n \geq 1$.

Remark

If $\mathfrak{q}$ is an ideal $\neq A$, then $\mathfrak{q}$ is primary if

$$xy \in \mathfrak{q} \Longrightarrow x \in \mathfrak{q} \text{ or } y \in r(\mathfrak{q}).$$

Primary Decomposition

Proposition (Proposition 4.1)

Let $\mathfrak{q}$ be a primary ideal in a ring A. Then $r(\mathfrak{q})$ is a prime ideal, and hence it's the smallest prime ideal containing $\mathfrak{p}$.

Definition

If $\mathfrak{p}$ is a prime ideal, then any primary ideal $\mathfrak{q}$ such that $r(\mathfrak{q}) = \mathfrak{p}$ is called $\mathfrak{p}$-*primary*.

Remark

Warning! If $\mathfrak{a}$ is an ideal whose radical $r(\mathfrak{a})$ is prime, then $\mathfrak{a}$ need not be primary (see slide 8).

Primary Decomposition

Example

The primary ideals in $\mathbb{Z}$ are (0) and (p^n) with p prime, since:

- They are primary ideals.
- They are the only ideals in $\mathbb{Z}$ with prime radical.

Primary Decomposition

Example

Let $A = k[x, y]$, k field, and $\mathfrak{q} = (x, y^2)$. Then $\mathfrak{q}$ is primary, since:

- $A/\mathfrak{q} \simeq \big(k[x, y]/(x)\big)/\big((x, y^2)/(x)\big) \simeq k[y]/(y^2)$.
- Every zero-divisor in $k[y]/(y^2)$ is a multiple of y, and hence is nilpotent in $k[y]/(y^2)$.

The radical of $\mathfrak{q}$ is $\mathfrak{p} = (x, y)$. We have

$$\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}.$$

This shows that a primary ideal need not be a prime-power.

Primary Decomposition

Example

Let $A = k[x, y, z]/(xy - z^2)$, and denote by $\overline{x}$, $\overline{y}$, $\overline{z}$ the images of x, y, z in A. Set $\mathfrak{p} = (\overline{x}, \overline{z})$.

- The ideal $\mathfrak{p}$ is prime, since $A/\mathfrak{p} \simeq k[x, y, z]/(x, z) \simeq k[y]$ is an integral domain.
- The ideal $\mathfrak{p}^2$ is not primary, since $\overline{xy} = \overline{z}^2 \in \mathfrak{p}^2$, but $\overline{x} \notin \mathfrak{p}^2$ and $\overline{y} \notin \mathfrak{p} = r(\mathfrak{p}^2)$.

This shows that a prime-power need be a primary ideal. However, we have the following result:

Proposition (Proposition 4.2)

If $\mathfrak{a}$ is an ideal in A whose radical radical $r(\mathfrak{a})$ is maximal, then $\mathfrak{a}$ is primary. In particular, every power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.

Lemma (Lemma 4.3)

If $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{q} = \cap_{i=1}^{n} \mathfrak{q}_i$ is $\mathfrak{p}$-primary.

Reminder (Ideal quotients; see Chapter 1)

- If $\mathfrak{a}$ is an ideal in A and $x \in A$, then

$$(\mathfrak{a} : x) = \{y \in A;\ xy \in \mathfrak{a}\}.$$

  In particular, $(\mathfrak{a} : x)$ is an ideal.
- $(\mathfrak{a} : x) \cap (\mathfrak{b} : x) = (\mathfrak{a} \cap \mathfrak{b} : x)$.

Lemma (Lemma 4.3)

Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal, and let $x \in A$. Then:

(i) *If $x \in \mathfrak{q}$, then $(q : x) = (1)$.*

(ii) *If $x \notin \mathfrak{q}$, then $(q : x)$ is $\mathfrak{p}$-primary. In particular, $r(\mathfrak{q} : x) = \mathfrak{p}$.*

(iii) *If $x \notin \mathfrak{p}$, then $(\mathfrak{q} : x) = \mathfrak{q}$.*

Primary Decomposition

Definition (Primary Decomposition)

- A *primary decomposition* of an ideal $\mathfrak{a}$ in A is of the form,

$$\mathfrak{a} = \bigcap_{1 \leq i \leq n} \mathfrak{q}_i, \qquad \mathfrak{q}_i \text{ primary ideal.}$$

- We say that the primary decomposition is *minimal* if
  (i) $r(\mathfrak{q}_i) \neq r(\mathfrak{q}_j)$ for $i \neq j$.
  (ii) $\mathfrak{q}_i \supsetneq \cap_{j \neq i} \mathfrak{q}_j$ for $i = 1, \ldots, n$.

Remarks

1. Some ideals don't admit a primary decomposition. An ideal that admits a primary decomposition is called *decomposable*.
2. Any primary decomposition can be reduced to a minimal one:
   - Lemma 4.3 allows us ro to reach (i) by taking the intersections of the $\mathfrak{q}_i$ with same radical.
   - Once we have (i) we achieve (ii) by throwing away the primary ideals $\mathfrak{q}_i$ such that $\mathfrak{q}_i \supseteq \cap_{j \neq i} \mathfrak{q}_j$.

Primary Decomposition

Theorem (1st Uniqueness Theorem; Theorem 4.5)

Let $\mathfrak{a}$ be a decomposable ideal and $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ a primary decomposition. Set $p_i = r(\mathfrak{q}_i)$, $i = 1, \ldots, n$. Then the $\mathfrak{p}_i$ are exactly the prime ideals of the form $r(\mathfrak{a} : x)$, $x \in A$. In particular, they don't depend on the primary decomposition of $\mathfrak{a}$.

Remarks

1. The proof of the uniqueness theorem shows that for each i there is $x_i \in A$ such that $r(\mathfrak{a} : x_i)$ is $\mathfrak{p}_i$-primary.
2. The primary components are not independent of the minimal decomposition in general (see next slide).

Example

Let $A = k[x, y]$, k field, and $\mathfrak{a} = (x^2, xy)$.

- Set $\mathfrak{p}_1 = (x)$ and $\mathfrak{p}_2 = (x, y)$. Then

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2$$

  This is a (minimal) primary decomposition, since:
  - $\mathfrak{p}_1$ is a prime ideal, and hence is primary.
  - $\mathfrak{p}_2$ is a maximal ideal, since $k[x, y]/(x, y) \simeq k$ is a field.
  - The ideal $\mathfrak{p}_2^2$ then is $\mathfrak{p}_2$-primary by Proposition 4.2.
- We have another minimal primary decomposition,

$$\mathfrak{a} = (x) \cap (x^2, y).$$

  Here (x^2, y) can be shown to be primary in the same way as in the example of slide 7.
- Thus, we may have distinct minimal primary decompositions.

Primary Decomposition

Definition

Let $\mathfrak{a}$ be a decomposable ideal in A and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ the prime ideals associated with any minimal primary decomposition.

- The prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are said to *belong to* $\mathfrak{a}$ or to be *associated with* $\mathfrak{a}$.
- The minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ are called *minimal* or *isolated* prime ideals associated with $\mathfrak{a}$.
- The other prime ideals are called *embedded*.

Remark

The terminology isolated/embedded comes from algebraic geometry (see Atiyah-MacDonald's book).

Example

Let $A = k[x, y]$ and $\mathfrak{a} = (x^2, xy) = (x) \cap (x, y)^2$. The only minimal prime ideal is (x), since $(x) \subset (x, y)$.

Primary Decomposition

Reminder (see Chapter 1)

- If $x \in \mathfrak{a}$, then $(0; x)$ is the annihilator of x, i.e.,
$$(0 : x) = \{y \in A;\ xy = 0\}.$$
- Let D be the set of all zero-divisors in A. By Lemma 1.15 we have
$$D = \bigcup_{x \neq 0} (0 : x) = \bigcup_{x \neq 0} r(0 : x).$$

Primary Decomposition

Proposition (Proposition 4.7)

Suppose that $\mathfrak{a}$ is a decomposable ideal in A. Let $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ be a minimal decomposition, and set $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Then, we have

$$\bigcup_{1 \leq i \leq n} \mathfrak{p}_i = \{x \in A;\ (\mathfrak{a} : x) \neq \mathfrak{a}\}.$$

In particular, if the zero ideal is decomposable, then the set D of zero-divisors is the union of the prime ideals that belong to 0.

Remark

Suppose that the zero ideal is decomposable. Let $0 = \cap_{i=1}^{n} \mathfrak{q}_i$ be a minimal decomposition, and set $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Let $\mathfrak{N} = r(0)$ be the nilradical of A. Then

$$\mathfrak{N} = r\big(\cap\, \mathfrak{q}_i\big) = \cap r(\mathfrak{q}_i) = \cap \mathfrak{p}_i.$$

That is, $\mathfrak{N}$ is the intersection of all prime ideals that belong to 0.

Primary Decomposition

Reminder (Rings of fractions; see Chapter 3)

Let S be a multiplicative subset of A.

- The ring of fractions $S^{-1}A$ consists of classes a/s with $a \in A$ and $s \in S$:

$$a/s = b/t \Longleftrightarrow \exists u \in S \text{ such that } u(at - bs) = 0.$$

- The map $f : a \to a/1$ is a ring homomorphism from A to $S^{-1}a$.
- The extension of an ideal $\mathfrak{a}$ is equal to $S^{-1}a$.

Notation

If $\mathfrak{a}$ is an ideal in A, we denote by $S(a)$ the contraction of $S^{-1}\mathfrak{a}$, i.e., $S(a) = f^{-1}(S^{-1}a)$.

Reminder (see Proposition 3.11)

- For any ideal $\mathfrak{a}$ in A we have
$$S(a) = \bigcup_{s \in S} (\mathfrak{a} : s).$$
- There is a one-to-one correspondance ($\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}$) between prime ideals of $S^{-1}A$ and prime ideals in A that don't meet S.
- $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = \left(S^{-1}\mathfrak{a}\right) \cap \left(S^{-1}\mathfrak{b}\right)$.
- $r\left(S^{-1}\mathfrak{a}\right) = S^{-1}r(\mathfrak{a})$.

Proposition (Proposition 4.8)

Let S be a multiplicative closed subset of A and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal of A.

(i) *If $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}\mathfrak{q} = S^{-1}A$.*

(ii) *If $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary and $S(\mathfrak{q}) = \mathfrak{q}$.*

In particular, we have a one-to-one correspondance ($\mathfrak{q} \leftrightarrow S^{-1}\mathfrak{q}$) between primary ideals of $S^{-1}A$ and primary ideals of A whose radicals do not meet S.

Primary Decomposition

Proposition (Proposition 4.9)

Let $\mathfrak{a}$ *be a decomposable ideal in* A *and* $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ *a primary decomposition. Set* $\mathfrak{p}_i = r(q_i)$, *and assume the indexation is so that* S *meets* $\mathfrak{p}_{m+1}, \ldots, \mathfrak{p}_n$, *but not* $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. *Then*

$$S^{-1}\mathfrak{a} = \bigcap_{1 \le i \le m} S^{-1}\mathfrak{q}_i, \qquad S(\mathfrak{a}) = \bigcap_{1 \le i \le m} \mathfrak{q}_i,$$

and these decompositions are minimal primary decompositions.

Primary Decomposition

Definition

Let $\mathfrak{a}$ be a decomposable idea. A set Σ of prime ideals belonging to $\mathfrak{a}$ is called *isolated* if, for any prime ideal $\mathfrak{p}'$ belonging to $\mathfrak{a}$, we have

$$\mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \Sigma \implies \mathfrak{p}' \in \Sigma.$$

Remark

If $\Sigma = \{\mathfrak{p}\}$, then Σ is isolated if and only if $\mathfrak{p}$ is minimal.

Reminder (see Proposition 3.11)

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals and $\mathfrak{a}$ an ideal in A. Then

$$\mathfrak{a} \subseteq \bigcup \mathfrak{p}_i \Longrightarrow \exists i \text{ such that } \mathfrak{a} \subseteq \mathfrak{p}_i.$$

Equivalently,

$$\mathfrak{a} \not\subseteq \mathfrak{p}_i \ \forall i \implies \mathfrak{a} \not\subseteq \bigcup \mathfrak{p}_i.$$

Primary Decomposition

Facts

Let $\mathfrak{a}$ be a decomposable ideal and Σ an isolated set of prime ideals belonging to $\mathfrak{a}$. Set

$$S = A \setminus \left(\bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p} \right) = \bigcap_{\mathfrak{p} \in \Sigma} (A \setminus \mathfrak{p}).$$

- S is a multiplicative closed subset of A, since this is the intersections of such subsets.
- Let $\mathfrak{p}'$ be a prime ideal belonging to $\mathfrak{a}$. If $\mathfrak{p}' \in \Sigma$, then $\mathfrak{p}' \cap S = \emptyset$.
- Moreover, by Proposition 3.11:

  $$\mathfrak{p}' \notin \Sigma \implies \mathfrak{p}' \not\subseteq \bigcup_{p \in \Sigma} \mathfrak{p} \implies \mathfrak{p}' \cap S \neq \emptyset.$$

  Thus,

  $$S \cap \mathfrak{p}' = \emptyset \iff \mathfrak{p}' \in \Sigma.$$

Theorem (2nd Uniqueness Theorem; Theorem 4.10)

Let $\mathfrak{a}$ be a decomposable ideal in A and $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ a primary decomposition. Set $\mathfrak{p}_i = r(\mathfrak{q}_i)$, and let $\Sigma = \{\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_m}\}$ be an isolated set of prime ideals belonging to $\mathfrak{a}$. Then $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_m}$ is independent of the decomposition.

In particular, we have:

Corollary (Corollary 4.11)

The isolated primary components (i.e., the components whose radicals are minimal primary ideals belonging to $\mathfrak{a}$) are uniquely determined by $\mathfrak{a}$.

Proof of 2nd Uniqueness Theorem.

Set $S = A \setminus \big(\cup_{\mathfrak{p} \in \Sigma} \mathfrak{p} \big) = A \setminus \big(\cup_{l=1}^{m} \mathfrak{p}_{i_l} \big)$.

- From slide 22 we know that S is multiplicatively closed and $\mathfrak{p}_i \cap S = \emptyset \Longleftrightarrow \mathfrak{p}_i \in \Sigma$.
- Applying Proposition 4.9 then gives

$$S(\mathfrak{a}) = \bigcap_{\mathfrak{p}_i \cap S = \emptyset} \mathfrak{q}_i = \bigcap_{l=1}^{m} \mathfrak{q}_{i_l}.$$

- Observe that S only depends on the prime ideals p_{i_l}. By the 1st uniqueness theorem these prime ideals only depends on $\mathfrak{a}$. Thus, $S(\mathfrak{a})$ depends only on $\mathfrak{a}$.
- It then follows that $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_m}$ only depends on $\mathfrak{a}$, and hence is independent of the decomposition.

The result is proved. □