Differentiable Manifolds
§4. Differential Forms on $\mathbb{R}^n$

Sichuan University, Fall 2020

Differential 1-Forms and the Differential of a Function

Definition (Cotangent Space)

The *cotangent space* to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ or $T_p^*\mathbb{R}^n$, is the dual space $T_p(\mathbb{R}^n)^\vee$ of the tangent space $T_p(\mathbb{R}^n)$.

Remark

In other words an element of $T_p^*(\mathbb{R}^n)$ is just a covector or linear functional on $T_p(\mathbb{R}^n)$.

Definition (Differential 1-Forms)

A *differential 1-form* (or *covector field*, or simply *1-form*) on an open subset $U \subset \mathbb{R}^n$ is a function ω that assigns to each $p \in U$ a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \longrightarrow \bigcup_{p \in U} T_p^*(\mathbb{R}^n), \qquad p \longrightarrow \omega_p \in T_p^*(\mathbb{R}^n).$$

Differential 1-Forms and the Differential of a Function

Definition (Differential of a Function)

The *differential* of a C^∞ function $f : U \to \mathbb{R}$ is the 1-form df defined by

$$(df)_p(v) = D_v f \qquad \text{for all } p \in U \text{ and } v \in T_p(\mathbb{R}^n).$$

Remarks

1. If $X = \sum a^j \frac{\partial}{\partial x^j}$ is any vector field on U, then

$$(df)_p(X_p) = X_p f = (Xf)(p) = \sum a^j(p) \frac{\partial f}{\partial x^j}(p).$$

2. We also denote by $df\big|_p$ the value of df at p.

Differential 1-Forms and the Differential of a Function

Example

- If $x^1, \dots, x^n$ are the coordinate functions, then

$$(dx^i)_p(v) = v^i \qquad \text{for any } v = \sum v^j \frac{\partial}{\partial x^j}\Big|_p \text{ in } T_p(\mathbb{R}^n).$$

- In particular, we have

$$(dx^i)_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta^i_j = \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases}$$

Proposition (Proposition 4.1)

$\{(dx^1)_p, \dots, (dx^n)_p\}$ is a basis of the cotangent space $T^*_p(\mathbb{R}^n)$. This is the dual basis of the basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ of the tangent space $T_p(\mathbb{R}^n)$.

Differential 1-Forms and the Differential of a Function

Fact

1. If ω is a 1-form on U, then, for every $p \in U$, we have a unique decomposition,
$$\omega_p = \sum a_i(p)(dx^i)_p, \qquad a_i(p) \in \mathbb{R}.$$
2. We write
$$\omega = \sum a_i dx^i,$$
where the coefficients a^i now are functions on U.

Definition

We say that the 1-form ω is C^∞ when all the coefficient functions $a_1, \ldots, a_n$ are all C^∞ on U.

Differential 1-Forms and the Differential of a Function

Proposition (Proposition 4.2; the differential in coordinates)

If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$df = \sum \frac{\partial f}{\partial x^i}.$$

Corollary

If f is a C^∞ function on U, then its differential df is a C^∞ 1-form.

Remark

The definition of $dx^1, \dots, dx^n$ as 1-forms gives a rigorous meaning to this notation in elementary calculus.

Differential k-Forms

Definition

A differential form of degree k (or a k-form) is a function ω that assigns to each $p \in U$ an an alternating k-linear form on the tangent space $T_p(\mathbb{R}^n)$, i.e., $\omega_p \in A_k(T_p(\mathbb{R}^n))$.

Remarks

1. As $A_1(T_p(\mathbb{R}^n)) = T_p^*(\mathbb{R}_n)$ this generalizes the notion of 1-form.
2. As $A_0(T_p(\mathbb{R}^n)) = \mathbb{R}$ a 0-form is merely a function on U.
3. There are no non-zero forms of degree $k > n$, since $\dim T_p(\mathbb{R}^n) = n$, and hence $A_k(T_p(\mathbb{R}^n)) = \{0\}$ for $k > n$.

Differential k-Forms

Example

If $I = (i_1, \ldots, i_k)$ is an ascending k-index $i_1 < \ldots < i_k$ the k-form dx^I is defined by

$$(dx^I)_p = dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}, \qquad p \in U.$$

Reminder (see Proposition 3.29)

At each point p the k-covectors dx_p^I form a basis of $A_k(T_p(\mathbb{R}^n))$ (*cf.* Proposition 3.29).

Differential k-Forms

Facts

1. If ω is a k-form, then, at each point p, we have a unique decomposition,

$$\omega_p = \sum a_I(p) dx_p^I, \qquad a_I(p) \in \mathbb{R},$$

   where the summation goes over all ascending k-indices.

2. We write

$$\omega = \sum a_I dx^I,$$

   where the coefficients a_I now are functions on U.

Definition

We say that the k-form is C^∞ on U when the coefficient functions a_I are all C^∞ on U.

Differential k-Forms

Definition

$\Omega^k(U)$ is the vector space of C^∞ k-forms on U.

Remark

$\Omega^0(U) = C^\infty(U)$, since 0-forms are functions.

Differential k-Forms

Definition (Wedge Product)

Given a k-form ω and ℓ-form τ, their *wedge product* is defined pointwise,

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p, \qquad p \in U.$$

Remarks

1. If we write $\omega = \sum a_I dx^I$ and $\tau = \sum b_J dx^J$, then
$$\omega \wedge \tau = \sum_{I,J} a_I b_J dx^I \wedge dx^J.$$
2. If I and J are not disjoints, then $dx^I \wedge dx^J = 0$. Thus,
$$\omega \wedge \tau = \sum_{I,J \text{ disjoint}} a_I b_J dx^I \wedge dx^J.$$

Differential k-Forms

Fact

- The wedge product is a bilinear map,

$$\wedge : \Omega^k(U) \times \Omega^\ell(U) \longrightarrow \Omega^{k+\ell}(U).$$

- This bilinear map is anticommutative and associative (*cf.* Proposition 3.21 and Proposition 3.25).
- As $\Omega^0(U) = C^\infty(U)$, for $k = 0$ the wedge product reduces to the pointwise multiplication of differential forms by functions,

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p.$$

Thus, if $f \in C^\infty(U)$ and $\omega \in \Omega^\ell(U)$, then $f \wedge \omega = f\omega$.

Differential k-Forms

Example

Let x, y, z be the coordinates in $\mathbb{R}^3$.

- The C^∞ 1-forms on $\mathbb{R}^3$ are

$$f dx + g dy + h dz, \qquad f, g, h \in C^\infty(\mathbb{R}^3).$$

- The C^∞ 2-forms on $\mathbb{R}^3$ are

$$f dx \wedge dy + g dx \wedge dz + h dy \wedge dz, \qquad f, g, h \in C^\infty(\mathbb{R}^3).$$

- The C^∞ 3-forms on $\mathbb{R}^3$ are

$$f dx \wedge dy \wedge dz, \qquad f \in C^\infty(\mathbb{R}^3).$$

Differential k-Forms

Facts

Define

$$\Omega^*(U) = \bigoplus_{k=0}^{\infty} \Omega^k(U) = \bigoplus_{k=0}^{n} \Omega^k(U).$$

1. With the wedge product as multiplication and the degree of a form as grading, $\Omega^*(U)$ is an anticommutative graded algebra.
2. With respect to the pointwise multiplication of functions, this is also a module over the ring $C^\infty(U)$.

Differential Forms as Multilinear Functions on Vector Fields

Definition

If ω is C^∞ 1-form and X is a C^∞ vector field on an open subset $U \subset \mathbb{R}^n$, the function $\omega(X)$ on U is defined by

$$\omega(X)_p = \omega_p(X_p), \qquad p \in U.$$

Fact

- In coordinates, if $\omega = \sum a_i dx^i$ and $X = \sum b^j \frac{\partial}{\partial x^j}$ with $a_i, b^j \in C^\infty(U)$, then

$$\omega(X) = \sum a_i b^i.$$

- This shows that $\omega(X)$ is a C^∞ function on U.

Fact

Let ω be a C^∞ 1-form on U.

- Given any function $f \in C^\infty(U)$ and any vector field $X \in \mathscr{X}(U)$, we have
$$\omega(fX) = f\omega(X).$$
- Set $\mathscr{F}(U) = C^\infty(U)$. Then the 1-form ω defines an $\mathscr{F}(U)$-linear map,
$$\mathscr{X}(U) \ni X \longrightarrow \omega(X) \in \mathscr{F}(U).$$

Fact

Similarly, any C^∞ k-form ω on U defines a k-linear map over $\mathscr{F}(U)$,

$$\underbrace{\mathscr{X}(U)\times\cdots\times\mathscr{X}(U)}_{k\text{ times}}\to\mathscr{F}(U),\quad (X_1,\ldots,X_k)\to\omega(X_1,\ldots,X_k),$$

where

$$\omega(X_1,\ldots,X_k)_p=\omega_p\left((X_1)_p,\ldots,(X_k)_p\right),\qquad p\in U.$$

Exterior Derivative

Definition (Exterior Derivative)

The *exterior derivative* of a C^∞ k-form on U is defined as follows:

- For $k = 0$ the exterior derivative of a 0-form (i.e., a C^∞ function) f on U is its differential,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

- For $k \geq 1$, the exterior derivative $\omega = \sum a_I dx^I \in \Omega^k(U)$ is

$$d\omega = \sum da_I \wedge dx^I = \sum_I \Big(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \Big) \wedge dx^I.$$

Remarks

- If $\omega \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$.
- In particular, $d\omega = 0$ for all $\omega \in \Omega^n(U)$.

Exterior Derivative

Example

Let $\omega = fdx + gdy$ be a 1-form on $\mathbb{R}^2$, where $f, g \in C^\infty(\mathbb{R}^2)$. Set $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Then

$$d\omega = (g_x - f_y) dx \wedge dy.$$

Exterior Derivative

Definition (Antiderivation of a Graded Algebra)

Let $A = \oplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$.

- An *antiderivation* of A is any linear map $D : A \rightarrow A$ such that

$$D(ab) = (Da)b + (-1)^k aDb \qquad \text{for all } a \in A^k \text{ and } b \in A.$$

- We say that D has *degree* m when $D(A^k) \subset A^{k+m}$ for all k.

Remark

- We can extend the grading to negative integers by setting $A^k = \{0\}$ for $k < 0$.
- This allows the degree m to be negative.

Exterior Derivative

Reminder

$\Omega^*(U) = \oplus\Omega^k(U)$ is a graded algebra over $\mathbb{R}$.

Proposition (Proposition 4.7)

The exterior derivative $d : \Omega^*(U) \to \Omega^*(U)$ *satisfies the following properties:*

(i) *It is an antiderivation of degree* 1, *i.e.,*

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

(ii) $d^2 = 0$, *i.e.,* $d(d\omega) = 0$ *for all* $\omega \in \Omega^*(U)$.

(iii) *If* $f \in C^\infty(U)$ *and* $X \in \mathscr{X}(U)$, *then* $(df)(X) = Xf$.

Proposition (Proposition 4.8)

The exterior derivative is the unique map $D : \Omega^*(U) \to \Omega^*(U)$ *that satisfies the properties (i)–(iii) above.*

Closed and Exact Forms

Definition

Let $\omega \in \Omega^k(U)$.

1. We say that ω is *closed* when $d\omega = 0$.
2. We say that ω is a *exact* when there is $\tau \in \Omega^{k-1}(U)$ such that $\omega = d\tau$.

Remarks

1. As $d(d\tau) = 0$ any exact k-form on U is closed. The converse may or may not hold depending on U.
2. We have

$$\omega \text{ is closed} \Longleftrightarrow \omega \in \ker d,$$

$$\omega \text{ is exact} \Longleftrightarrow \omega \in \operatorname{ran} d.$$

Closed and Exact Forms

Example (see Exercise 4.9)

Consider the following 1-form on $\mathbb{R}^2 \setminus 0$,

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Then ω is closed, but it is not exact.

Remark

If $f \in C^\infty(\mathbb{R}^2 \setminus 0)$ is such that $df = \omega$, then it can be shown that

$$\frac{d}{dt}f(\cos t, \sin t) = 1.$$

This implies that $\int_0^{2\pi} \frac{d}{dt}f(\cos t, \sin t)dt = 2\pi \neq 0$, which is absurd.

Closed and Exact Forms

Theorem (Poincaré Lemma; see Corollary 27.13)

If U is star shaped about a point and $k \geq 1$, then every closed form $\omega \in \Omega^k(U)$ is exact.

Remarks

1. In particular, any closed k-form on $\mathbb{R}^n$ or an open ball with $k \geq 1$ is exact.
2. Poincaré Lemma is a special case of a more general result for "contractible manifolds" (see Section 27 of Tu's book).
3. A direct proof of Poincaré Lemma can be found in the book *Introduction to Smooth Manifolds* by John M. Lee.

Closed and Exact Forms

Definition (Cochain Complexes)

- A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ together with linear maps $d_k : V^k \rightarrow V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *cochain complex*,

$$0 \longrightarrow V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} \cdots V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} \cdots$$

- Its *cohomology space of degree k* is the quotient space,

$$H^k(V) = \frac{\ker d_k}{\operatorname{ran} d_{k-1}}.$$

Remark

By convention $d_{-1} = 0$, and so $\operatorname{ran} d_0 = \{0\}$ and $H^0(V) = \ker d_0$.

Closed and Exact Forms

Definition (de Rham Cohomology)

- If U is an open subset of $\mathbb{R}^n$, then the cochain complex,

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots$$

  is called the *de Rham complex* of U.
- Its cohomology spaces are called *de Rham cohomology spaces* of U and are denoted by $H^k(U)$.

Remark

$$H^0(U) = \ker\{d : \Omega^0(U) \to \Omega^1(U)\} = \{f \in C^\infty(U);\ df = 0\}.$$

For $k \geq 1$, we have

$$H^k(U) = \frac{\ker\{d : \Omega^k(U) \to \Omega^{k+1}(U)\}}{\operatorname{ran}\{d : \Omega^{k-1}(U) \to \Omega^k(U)\}} = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

Closed and Exact Forms

Fact

In terms of de Rham cohomology, Poincaré Lemma means that if U is star shaped, then

$$H^k(U) = \{0\} \qquad \text{for all } k \geq 1.$$

Remark

- It can be shown that the de Rham cohomology of U depends only on its *topology*, or even its "homotopy type" (see Section 27 and Lee's book).
- It's an instance of applying "differential" techniques to study topology (differential topology).

Closed and Exact Forms

Example

If f is a C^∞ function on U, then

$$df = 0 \Longleftrightarrow \sum \frac{\partial f}{dlx^i} dx^i = 0 \Longleftrightarrow \frac{\partial f}{dlx^1} = \cdots = \frac{\partial f}{dlx^n} = 0.$$

Thus, $df = 0$ if and only if f is constant on each connected component of U.

Fact

For every open subset $U \subset \mathbb{R}^n$,

$$H^0(U) \simeq \mathbb{C}^m,$$

where m is the number of connected components of U .

Remark

One of the goal of this course is the generalization of de Rham cohomology to manifolds (see Sections 24–29).

Definition

A *vector valued function* on an open subset $U \subset \mathbb{R}^3$ is a function,

$$\mathbf{F} = \langle P, Q, R \rangle : U \longrightarrow \mathbb{R}^3.$$

Remark

A vector valued function assigns to each $p \in U$ a vector

$$\mathbf{F}_p \in \mathbb{R}^3 \simeq T_p(\mathbb{R}^3)$$

Therefore, a vector valued function on U can also be thought of as a vector field on U.

Applications to Vector Calculus

Reminder

Gradient, curl and divergence are operators on scalar-valued functions and vector-valued functions,

$$\{\text{scal. func.}\} \xrightarrow{\text{grad}} \{\text{vect. func.}\} \xrightarrow{\text{curl}} \{\text{vect. func.}\} \xrightarrow{\text{div}} \{\text{scal. func.}\},$$

$$\operatorname{grad} f = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix},$$

$$\operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q - z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix},$$

$$\operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = P_x + Q_y + R_z.$$

Applications to Vector Calculus

Facts

1. We identify 1-forms and vector fields on U via

$$Pdx + Qdy + Rdz \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

2. Under this identification, for any $f \in C^\infty(U)$, we have

$$df = f_x dx + f_y dy + f_z dz \longleftrightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \operatorname{grad} f.$$

Facts

1. We also identify 2-forms with vector fields,

$$P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

2. For any 1-form $\omega = Pdx + Qdy + Rdz$, we have

$$d\omega = (R_y - Q_z) dy \wedge dz - (R_x - P_z) dz \wedge dx + (Q_x - P_y) dx \wedge dy.$$

Thus,

$$d\omega \longleftrightarrow \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix} = \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

Facts

1. We identify 3-forms with functions

$$f dx \wedge dy \wedge dz \longleftrightarrow f$$

2. For any 2-form $\omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$ we have

$$d\omega = (P_x + Q_y + R_z) dx \wedge dy \wedge dz.$$

Thus,

$$d\omega \longleftrightarrow P_x + Q_y + R_z = \operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

Summary

If U is an open subset of $\mathbb{R}^3$, the identification of differential forms with functions and vector fields the exterior derivative corresponds to the operators grad, curl and div:

$$\begin{array}{ccccccc}
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
C^\infty(U) & \xrightarrow{\text{grad}} & \mathscr{X}(U) & \xrightarrow{\text{curl}} & \mathscr{X}(U) & \xrightarrow{\text{div}} & C^\infty(U).
\end{array}$$

Consequence

If U is an open subset of $\mathbb{R}^3$, then the equality $d \circ d$ on $\Omega^0(U)$ and $\Omega^1(U)$ translates into the following:

Proposition (Proposition A)

For every function $f \in C^\infty(U)$, we have

$$\operatorname{curl}(\operatorname{grad} f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Proposition (Proposition B)

For every vector field $\mathbf{F} \in \mathscr{X}(U)$, we have

$$\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0.$$

Applications to Vector Calculus

Consequence

- If U is an open subset of $\mathbb{R}^3$, then a C^∞ vector field $\mathbf{F} = \langle P, Q, R \rangle$ on U is the gradient of a function $f \in C^\infty(U)$ if and only the corresponding 1-form $Pdx + Qdy + Rdz$ is df.
- Therefore, Poincaré Lemma for 1-forms translates into:

Proposition (Proposition C)

Assume that U is a star shaped open subset of $\mathbb{R}^3$. Then a C^∞ vector field $\mathbf{F}$ on U is the gradient of a function $f \in C^\infty(U)$ if and only if $\operatorname{curl} \mathbf{F} = 0$.

Convention on Subscripts and Superscripts

Convention (Covectors)

- Coordinates $x^1, \dots, x^n$ and multicovectors/forms $\omega^1, \dots, \omega^k$ are indexed by *superscripts*.
- The coordinates of k-forms with respect to the basis $\{dx^I\}$ are index by *subscripts*,

$$\omega = \sum a_i dx^i, \qquad \omega = \sum a_I dx^I.$$

- The subscripts in a_i or a_I "cancel out" the superscripts in dx^i or dx^I.

Convention on Subscripts and Superscripts

Convention (Vectors)

- Coordinate vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are considered to have subscripts since the index i in $\frac{\partial}{\partial x^i}$ is in the lower half of the fraction.
- Vectors $v_1, \dots, v_k$ and vector fields $X_1, \dots, X_k$ are indexed by *subscripts*.
- Coordinates of a vector v in a given basis $\{e_i\}$ or a vector field in the basis $\{\frac{\partial}{\partial x^i}\}$ are indexed by *superscripts*,

$$v = \sum v^i e_i, \qquad X = \sum X^i \frac{\partial}{\partial x^i}.$$

- The superscripts in v^i or X^i "cancel out" the subscripts in e_i or $\frac{\partial}{\partial x^i}$.