Differentiable Manifolds
§3. The Exterior Algebra of Multicovectors

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Dual Space

Definition

1. If V and W is are vector spaces we denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$.
2. The *dual space* $\mathrm{Hom}(V, \mathbb{R})$ is denoted by $V^\vee$. It consists of all linear forms $f : V \to \mathbb{R}$.
3. The elements of $V^\vee$ are called *covectors*.

Convention

We assume that V is a *finite-dimensional* space with basis $e_1, \ldots, e_n$.

Fact

Every $v \in V$ can be uniquely written as $v = \sum v^i e_i$ with $v^i \in \mathbb{R}$.

Dual Space

Definition

The *i-coordinate function* $\alpha^i : V \to \mathbb{R}$ is defined by

$$\alpha^i(v) = v^i \qquad \text{if } v = \sum v^j e_j.$$

Remark

We have

$$\alpha^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Here δ^i_j is called the *Kronecker symbol*.

Dual Space

Proposition (Proposition 3.1)

The coordinate functions $\alpha^1, \ldots, \alpha^n$ *form a basis of* $V^\vee$*.*

Definition

The basis $\alpha^1, \ldots, \alpha^n$ is called the *dual basis* of the basis $e_1, \ldots, e_n$.

Corollary (Corollary 3.2)

The dual space $V^\vee$ *has the same dimension as* V*, and hence has finite dimension.*

Permutations

Definition (Permutations)

1. A *permutation* is any bijection of $\{1, \ldots, k\}$ onto itself.
2. The set of all permutations of $\{1, \ldots, k\}$ is denoted by S_k.

Remarks

1. In other words a permutation $\sigma \in S_k$ is a *reordering* of the set $\{1, \ldots, k\}$.
2. $|S_k| = k!$ (number of elements of S_k).
3. It is sometimes convenient to represent a permutation $\sigma \in S_k$ by its matrix,

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

Permutations

Facts

1. The composition of maps induces a group law on S_k,

$$(\sigma\tau) = \sigma \circ \tau, \qquad \sigma, \tau \in S_k$$

2. The identity permutation is the identity element of S_k.

Definition

S_k is called the *symmetric group* of degree k.

Permutations

Definition (Transposition)

The *transposition* $\tau = (a, b)$, $a \neq b$, is the permutation that exchanges a and b and leaves all other elements unchanged, i.e.,

$$\tau(j) = \begin{cases} b & \text{for } i = a, \\ a & \text{for } i = b, \\ i & \text{otherwise.} \end{cases}$$

Example

The transposition $(1, 2) \in S_3$ has matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}.$$

Permutations

Definition (r-Cycles)

An r*-cycle* $\sigma = (a_1, \ldots, a_r)$, where the a_i are distinct, is the permutation such that

1. $\sigma(a_i) = a_{i+1}$ for $i = 1, \ldots, r-1$.
2. $\sigma(a_r) = a_1$.
3. It leaves all other elements unchanged.

Example

The 3-cycle $(123) \in S_4$ has matrix,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

Remarks

1. Any 1-cycle (a) is the identity permutation.
2. 2-cycles are just transpositions.

Permutations

Example (The symmetric group S_2)

The symmetric group S_2 consists of

- The identity (1).
- The transposition $(1, 2)$.

Example (The symmetric group S_3)

The symmetric group S_3 consists of

- The identity (1).
- Transpositions: $(1, 2)$, $(1, 3)$, and $(2, 3)$.
- 3-cycles: (123) and (132).

Permutations

Example (The symmetric group S_4)

The symmetric group S_4 consists of

- The identity (1).
- Transpositions: $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,4)$.
- 3-cycles: (123), (124), (134), (234), (132), (142), (143), (243).
- 4-cycles: (1234), (1324), (1432), (1243), (1342), (1423).
- Products of 2-cycles: (12)(34), (13)(24), (14)(23).

Fact

Any permutation $\sigma \in S_k$ can be written as a product of *disjoint* cycles.

Permutations

Facts

1. Any permutation $\sigma \in S_k$ can be written as a product of transpositions,
$$\sigma = \tau_1 \tau_2 \cdots \tau_n.$$
2. The decomposition is not unique, but the parity of n depends only on σ.

Definition

We say that σ is *even* (resp., *odd*) when n is even (resp., odd).

Permutations

Example

We have

$$\begin{aligned}(1234) &= (14)(13)(12)\\ &= (12)(24)(23)\\ &= (23)(14)(12)(13)(23).\end{aligned}$$

Fact

Any r-cycle $(a_1, \ldots, a_r)$ can be decomposed as the product of $r-1$ transpositions,

$$(a_1 a_r)(a_1 a_{r-1}) \cdots (a_1 a_2).$$

Therefore, it has the opposite parity of r.

Permutations

Definition (Sign of a Permutation)

The *sign* of a permutation $\sigma \in S_k$ is

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Fact

The map $\operatorname{sgn} : S_k \to \{\pm 1\}$ *is a morphism of groups, i.e.,*

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau), \qquad \operatorname{sgn}((1)) = 1.$$

Consequence

If $\sigma \in S_k$ is the product of p cycles of even length and q cycles of odd length, then

$$\operatorname{sgn}(\sigma) = (-1)^p.$$

Multilinear Functions

Definition

Set $V^k = V \times \cdots \times V$, where V is a vector space.

1. A function $f : V^k \to \mathbb{R}$ is *k-linear* when it is linear in each of its arguments, i.e.,

$$f(v_1, \ldots, \lambda v_i + \mu w_i, \cdots, v_n) = \lambda f(v_1, \ldots, v_i, \cdots, v_n) + \mu f(v_1, \ldots, w_i, \cdots, v_n).$$

2. The space of k-linear maps on V is denoted by $L_k(V)$.

Remarks

1. A 2-linear function is called *bilinear*.
2. A k-linear map is also called a *k-tensor* or *tensor of degree k*.

Multilinear Functions

Example

The dot product on $\mathbb{R}^n$ is bilinear,

$$v \cdot w = \sum v^j w^j, \qquad \text{where } v = \sum v^j e_j, \; w = \sum w^j e_j.$$

Here $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$.

Example

The determinant of vectors in $\mathbb{R}^n$ is n-linear,

$$(v_1, \ldots, v_n) \longrightarrow \det[v_1 \cdots v_n].$$

Multilinear Functions

Definition

Let $f : V^k \to \mathbb{R}$ be a k-linear function.

1. We say that f is *symmetric* when
$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k) \qquad \text{for all } \sigma \in S_k.$$
2. We say that f is *alternating* when
$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \dots, v_k) \qquad \text{for all } \sigma \in S_k.$$

Remark

If $f : V^2 \to \mathbb{R}$ is bilinear, then

$$\begin{aligned} f \text{ is symmetric} \quad &\Longleftrightarrow \quad f(v,w) = f(u,w) \quad \forall v,w \in V, \\ f \text{ is alternating} \quad &\Longleftrightarrow \quad f(v,w) = -f(u,w) \quad \forall v,w \in V. \end{aligned}$$

Multilinear Functions

Fact

Let $f : V^k \to \mathbb{R}$ be k-linear. TFAE:

(i) f is symmetric.

(ii) $f(v_1, \ldots, v_n)$ remains unchanged whenever two arguments are interchanged, i.e.,

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

Fact (Problem 3.4 + Problem 3.5)

Let $f : V^k \to \mathbb{R}$ be k-linear. TFAE:

(i) f is alternating.

(ii) $f(v_1, \ldots, v_n)$ changes sign whenever two arguments are interchanged, i.e.,

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

(iii) $f(v_1, \ldots, v_n) = 0$ whenever two arguments agree.

Multilinear Functions

Fact

More generally, it can be shown that if $f : V^k \to \mathbb{R}$ is an alternating k-linear function, then

$$f(v_1, \ldots, v_n) = 0 \quad \text{whener } v_1, \ldots, v_n \text{ are linearly dependent.}$$

Consequence

Any alternating k-linear function on V with $k > \dim V$ must be zero.

Multilinear Functions

Examples

1. The dot product $v \cdot w$ on $\mathbb{R}^n$ is symmetric.
2. The determinant of vectors $\det[v_1 \cdots v_n]$ on $\mathbb{R}^n$ is alternating.
3. The cross product $v \times w$ on $\mathbb{R}^3$ is alternating.

Example

Let $f, g : V \to \mathbb{R}$ be linear maps, and define their wedge product $f \wedge g : V^2 \to \mathbb{R}$ by

$$(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u).$$

Then $f \wedge g$ is an alternating bilinear map.

Multilinear Functions

Definition

Let V be a vector space.

1. $A_k(V)$ is the space of all alternating k-linear maps $f : V^k \rightarrow \mathbb{R}$.
2. Elements of $A_k(V)$ are called *alternating k-tensors*, *k-covectors*, or *multicovectors of degree k*.

Remarks

1. By convention a 0-covector is a scalar, so that $A_0(V) = \mathbb{R}$.
2. A 1-covector is just a covector, and so $A_1(V) = V^\vee$.
3. $A_k(V) = \{0\}$ for $k > \dim V$.

Permutation Action on Multilinear Functions

Definition

Given a k-linear function on V and a permutation $\sigma \in S_k$, we define the k-linear function σf by

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \qquad v_i \in V.$$

Remark

$$\begin{aligned} f \text{ is symmetric} \quad &\iff \quad \sigma f = f \quad \forall \sigma \in S_k, \\ f \text{ is alternating} \quad &\iff \quad \sigma f = \operatorname{sgn}(\sigma) f \quad \forall \sigma \in S_k. \end{aligned}$$

Permutation Action on Multilinear Functions

Lemma (Lemma 3.11)

Let $f : V^k \to \mathbb{R}$ be k-linear. Then

$$\tau(\sigma f) = (\tau\sigma) f \qquad \forall \sigma, \tau \in S_k.$$

Definition

Given a set X and a group G with identity e, a *left action* is a map $G \times X \ni (g, x) \to gx \in X$ such that

1. $ex = x$ for all $x \in X$.
2. $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Fact

The map $(\sigma, f) \to \sigma f$ is a left action of the symmetric group S_k on the vector space $L_k(V)$ of k-linear functions.

Remark

Each permutation $\sigma \in S_k$ acts linearly on $L_k(V)$.

Permutation Action on Multilinear Functions

Remark

1. We also define a *right action* as a map $X \times G \ni (x, g) \to xg \in X$ such that
   1. $ex = x$ for all $x \in X$.
   2. $(xg)h = x(gh)$ for all $g, h \in G$ and $x \in X$.
2. If $(g, x) \to gx$ is a left action, then $(x, g) \to xg^{-1}$ is a right action.

Symmetrizing and Alternating Operators

Definition (Symmetrizing Operator)

The *symmetrizing operator* $S : L_k(V) \to L_k(V)$ is defined by

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

That is,

$$(Sf)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right).$$

Definition (Alternating Operator)

The *alternating operator* $S : L_k(V) \to L_k(V)$ is defined by

$$Sf = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma f.$$

Symmetrizing and Alternating Operators

Proposition (Proposition 3.12)

Let $f : V^k \to \mathbb{R}$ be k-linear. Then

1. *Sf is symmetric.*
2. *Af is alternating.*

Lemma (Lemma 3.14)

If $f : V^k \to \mathbb{R}$ is k-linear and alternating, then $Af = (k!)f$.

Remark

Let $f : V^k \to \mathbb{R}$ be k-linear. Then

$$\begin{aligned} f \text{ is symmetric} &\iff Sf = (k!)f, \\ f \text{ is alternating} &\iff Af = (k!)f. \end{aligned}$$

The Tensor Product

Definition

Let $f : V^k \to \mathbb{R}$ be k-linear and let $g : V^\ell \to \mathbb{R}$ be ℓ-linear. Their *tensor product* is the $(k + l)$-linear function $f \otimes g : V^{k+\ell} \to \mathbb{R}$ defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}), \quad v_i \in V.$$

Example

Let $e_1, \dots, e_n$ be a basis of V and $\alpha^1, \dots, \alpha^n$ its dual basis. Given a bilinear map $g : V \times V \to \mathbb{R}$, set $g_{ij} = g(e_i, e_j)$. Then

$$g = \sum g_{ij}\alpha^i \otimes \alpha^j.$$

The Tensor Product

Exercise (Exercise 3.17, Associativity of the tensor product)

If f, g and h are multilinear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

The Wedge Product

Definition

Given alternating functions $f \in A_k(V)$ and $g \in A_\ell(V)$, their *wedge product* (or *exterior product*) is the alternating function $f \wedge g \in A_{k+\ell}(V)$ defined by

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

That is,

$$(f \wedge g)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).$$

The Wedge Product

Example (Example 3.19; wedge product of covectors)

If $f \in A_1(V)$ and $g \in A_2(V)$, then

$$(f \wedge g)(v_1, v_2) = f(v_1)g(v_2) - f(v_2)g(v_2).$$

Example (Example 3.18)

If $f \in A_2(V)$ and $g \in A_1(V)$, then

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1).$$

The Wedge Product

Definition

A permutation $\sigma \in S_{k+\ell}$ is a called a (k, ℓ)-*shuffle* when

$$\sigma(1) < \cdots < \sigma(k) \qquad \text{and} \qquad \sigma(k+1) < \cdots < \sigma(k+\ell).$$

Fact

It can be shown that if $f \in A_k(V)$ and $g \in A_\ell(V)$, then

$$(f \wedge g)(v_1, \ldots, v_{k+\ell}) = \sum_{\substack{(k,\ell)\text{-shuflles} \\ \sigma}} \operatorname{sgn}(\sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).$$

Anticommutativity of the Wedge Product

Proposition (Proposition 3.21)

The wedge product is anticommutative, i.e., if $f \in A_k(V)$ *and* $g \in A_\ell(V)$*, then*

$$g \wedge f = (-1)^{k\ell} f \wedge g.$$

Corollary (Corollary 3.23)

If $f \in A_k(V)$ *and* k *is odd, then* $f \wedge f = 0$.

Associativity of the Wedge Product

Lemma (Lemma 3.24)

Let $f \in A_k(V)$ and $g \in A_\ell(V)$. Then

(i) $A(A(f) \otimes g) = k!A(f \otimes g)$.

(ii) $A(f \otimes A(g)) = \ell!A(f \otimes g)$.

Proposition (Proposition 3.25; Associativity of the wedge product)

Let $f \in A_k(V)$, $g \in A_\ell(V)$, and $h \in A_m(V)$. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Consequence

We may omit parantheses in wedge products and write $(f \wedge h) \wedge h$ and $f \wedge (g \wedge h)$ simply as $f \wedge g \wedge h$.

Associativity of the Wedge Product

Corollary (Corollary 3.26; corollary of the proof of Proposition 3.25)

If $f \in A_k(V)$, $g \in A_\ell(V)$, and $h \in A_m(V)$, then

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} A(f \otimes g \otimes h).$$

Fact

More generally, it can be shown by induction that if $f_i \in A_{k_i}(V)$, $i = 1, \ldots, r$, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{k_1! \cdots k_r!} A(f_1 \otimes \cdots \otimes f_r).$$

Associativity of the Wedge Product

Proposition (Proposition 3.27; wedge product of 1-covectors)

If $\alpha^1, \ldots, \alpha^k$ are in $A_1(V) = V^\vee$ and $v_1, \ldots, v_n$ are vectors in V, then

$$\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(v_1, \ldots, v_k) = \det\left[\alpha^i(v_j)\right].$$

Associativity of the Wedge Product

Definition (Graded Algebras)

1. An algebra A over a field $\mathbb{K}$ called *graded* when it can be decomposed as
$$A = \bigoplus_{k=0}^{\infty} A^k,$$
where the A^k are subspaces such that the multiplication maps $A^k \times A^\ell$ to $A^{k+\ell}$.

2. We then say that A is *anticommutative* (or *graded commutative*) when
$$ba = (-1)^{k\ell} ab \qquad \text{for all } a \in A^k \text{ and } b \in A^\ell.$$

Associativity of the Wedge Product

Definition

Given graded algebras $A = \bigoplus A^k$ and $B = \bigoplus B^k$, a *graded algebra homomorphism* $f : A \to B$ is an algebra homomorphism that preserves the degree, i.e.,

$$f(A^k) \subset B^k \qquad \text{for all } k \geq 0.$$

Example

- The polynomial algebra $A = \mathbb{R}[x, y]$ is graded by degree.
- Here A^k consists of homogeneous polynomials of total degree k, i.e., linear combinations of monomials $x^p y^q$ with $p + q = k$.

Associativity of the Wedge Product

Definition (Exterior Algebra)

If V is vector space of dimension n, its *exterior algebra* (or *Grassmann algebra*) is the vector space,

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V).$$

Fact

With respect to the wedge product $A_*(V)$ is an anticommutative graded algebra.

Basis of k-Covectors

Convention

- V is an n-dimensional vector space with basis $e_1, \ldots, e_n$.
- $\alpha^1, \ldots, \alpha^n$ is the dual basis of $V^\vee$.

Definition

Given any multi-indice $I = (i_1, \ldots, i_k)$ we define

$$e_I = (e_{i_1}, \ldots, e_{i_k}) \qquad \text{and} \qquad \alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

Remark

Here $e_I \in V^k$ and $\alpha^I \in A_k(V)$.

Basis of k-Covectors

Observation

By multi-linearity a k-linear function $f : V^k \to \mathbb{R}$ is completely determined by its values on the k-tuples $(e_{i_1}, \ldots, e_{i_k})$.

Facts

Let $f \in A_k$ and let $(i_1, \ldots, i_k)$ be a multi-index. Then

1. $f(e_{i_1}, \ldots, e_{i_k}) = 0$ whenever $i_p = i_q$ for some $p \neq q$.
2. If all the i_p are distincts, then there is a unique permutation $\sigma \in S_k$ such that $i_{\sigma(1)} < \ldots < i_{\sigma(k)}$.
3. We then have
$$f(e_{i_1}, \ldots . e_{i_k}) = \mathrm{sgn}(\sigma) f(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(k)}}).$$

Consequence

If $f \in A_k(V)$, then its completely determined by its values on k-uples of the form,
$$(e_{i_1}, \ldots, e_{i_k}) \qquad \text{with } 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

Basis of k-Covectors

Lemma (Lemma 3.28)

Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ be strictly ascending multi-indices, i.e., $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$. Then

$$\alpha^I(e_J) = \delta^I_J = \begin{cases} 1 & \text{for } I = J, \\ 0 & \text{for } I \neq J. \end{cases}$$

Proposition (Proposition 3.29)

The k-covectors α^I, $I = (i_1 < \ldots < i_k)$, form a basis of $A_k(V)$.

Corollary (Corollary 3.30 + Corollary 3.31)

1. *If $0 \leq k \leq n$, then $\dim A_k(V) = \binom{n}{k}$.*
2. *If $k > n$, then $A_k(V) = \{0\}$.*