Differentiable Manifolds
§2. Tangent Vectors in $\mathbb{R}^n$ as Derivations

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Directional Derivative

Remark

- A vector at a point p in $\mathbb{R}^3$ can be visualize as an arrow emanating from p.
- It can also be represented as a column,

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}.$$

Definition (Tangent Space)

- The tangent space $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the vector space of all arrows emanating from p.
- Elements of $T_p(\mathbb{R}^n)$ are called tangent vectors (or simply vectors).

Remarks

1. We identify $T_p(\mathbb{R}^n)$ with the space of n-columns, and hence tangent vectors are identified with n-columns.
2. We sometime write $T_p\mathbb{R}^n$ for $T_p(\mathbb{R}^n)$.

Directional Derivative

Convention

- We shall denote a point in $\mathbb{R}^n$ as $p = (p^1, \ldots, p^n)$ and a tangent vector in $T_p(\mathbb{R}^n)$ as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{or} \quad v = \langle v^1, \ldots, v^n \rangle.$$

- We usually denote the canonical basis of $\mathbb{R}^n$ or $T_p(\mathbb{R}^n)$ by $e_1, \ldots, e_n$, so that $v = \sum v^j e_j$.

Directional Derivative

Definition

Let f be a C^∞-function on a neighbrohood of $p = (p^1, \ldots, p^n)$, and let $v = \langle v^1, \ldots, v^n \rangle$ be a tangent vector. The directional derivative of f in the direction of v at p is defined to be

$$D_v f = \frac{d}{dt}\bigg|_{t=0} f(p + tv).$$

Remarks

1. In other words $D_v f = \frac{d}{dt}\big|_{t=0} f(c(t))$, where $c(t) = p + tv$.
2. In the notation $D_v f$ it is implicitly understood that we evaluate at p, since v is a tangent vector at p. Thus, $D_v f$ is a number, not a function.

Directional Derivative

Fact

By using the Chain Rule we find that

$$D_v f = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

Definition

We write

$$D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p$$

for the map that assigns to any C^∞-function f near p its directional derivative $D_v f$.

Remark

As we shall see, the assignment $v \rightarrow D_v$ provides us with an alternative description of tangent vectors at p.

Germs of Functions

Observation

Two C^∞ functions that agree on a neighborhood of p have the same directional derivatives at p. Therefore, it is natural to declare such functions to be equivalent.

Germs of Functions

Definition (Relation)

Let S be a set.

1. A relation on S is given by a subset $R \subset S \times S$.
2. Given $x, y \in S$ we write $x \sim y$ when $(x, y) \in R$.

Definition (Equivalence Relation)

- The relation R is an equivalence relation when it satisfies the following properties:
  - (i) Reflexivity: $x \sim x$ for all $x \in S$.
  - (ii) Symmetry: If $x \sim y$, then $y \sim x$.
  - (iii) Transitivity: If $x \sim y$ and $y \sim z$, then $x \sim z$.
- If $x \sim y$, then we say that x and y are equivalent.
- The set of all $y \in S$ that are equivalent to x is called the equivalence class of x.

Germs of Functions

Definition

We define a relation on C^∞-functions near p at follows:

- S is a set of pairs (f, U), where U is a neighborhood of p and f is a C^∞-function on U.
- The relation on S is given by

$$(f, U) \sim (g, V) \iff f = g \text{ near } p.$$

Fact

This defines an equivalence relation.

Definition

1. The equivalence class of (f, U) is called the germ of f at p.
2. The set of all germs at p is denoted by $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞.

Germs of Functions

Example

The functions

$$f(x) = \frac{1}{1-x}, \qquad x \neq 1,$$
$$g(x) = 1 + x + x^2 + \cdots, \qquad |x| < 1,$$

have the same germ at any point of the interval $(-1, 1)$.

Definition (Algebra over a Field)

An algebra over a field $\mathbb{K}$ is a vector space A equipped with an associative multiplication $(a, b) \to ab$ that is compatible with scalar multiplication and addition of vectors. That is, it satisfies the following properties:

(i) Associativity: $a(bc) = (ab)c$.

(ii) Distributivity: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

(iii) Homogeneity: $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in \mathbb{K}$.

Remark

Equivalently, an algebra A over $\mathbb{K}$ is a ring equipped with a scalar multiplication that satisfies (iii) and turns A into a vector space.

Definition (Algebra Homomorphism)

Given algebras A and A' over $\mathbb{K}$, an algebra homomorphism $L : A \to A'$ is any $\mathbb{K}$-linear map that is multiplicative, i.e.,

$$L(ab) = L(a)L(b) \qquad \forall a, b \in A.$$

Fact

Let p be a point in $\mathbb{R}^n$.

1. The addition, scalar multiplication and multiplication of functions induces corresponding operations on the set of germs C_p^∞ (see Problem 2.2).
2. This turns C_p^∞ into an algebra over $\mathbb{R}$.

Derivations at a Point

Facts

Let p be a point in $\mathbb{R}^n$ and v a tangent vector at p.

1. The directional derivative gives rise to a map,

$$D_v : C_p^\infty \longrightarrow \mathbb{R}.$$

2. This map is $\mathbb{R}$-linear and satisfies Leibniz's Rule:

$$D_v(fg) = f(p)D_v(g) + (D_v f)g(p).$$

Derivations at a Point

Definition

1. Any linear map $D : C_p^\infty \to \mathbb{R}$ that satisfies Leibniz's Rule is called a derivation at p (or a point-derivation of C_p^∞).
2. The set of all derivations at p is denoted by $\mathscr{D}_p(\mathbb{R}^n)$.

Fact

$\mathscr{D}_p(\mathbb{R}^n)$ is a vector space over $\mathbb{R}$.

Lemma (Lemma 2.1)

Let $D : C_p^\infty \to \mathbb{R}$ be a derivation at p. Then $D(c) = 0$ for every constant function c.

Derivations at a Point

Theorem (Theorem 2.2)

Let $\phi : T_p(\mathbb{R}^n) \to \mathscr{D}_p(\mathbb{R}^n)$ be the map defined by

$$\phi(v) = D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p, \qquad v = \langle v^1, \ldots, v^n \rangle \in T_p(\mathbb{R}^n).$$

Then ϕ is a linear isomorphism.

Consequence

- This isomorphism allows us to identify tangent vectors at p with derivation at p.
- Under this identification,

$$\begin{aligned} \text{Canonical basis } e_1, \ldots, e_n \quad &\longleftrightarrow \quad \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p, \\ v = \langle v^1, \ldots, v^n \rangle = \sum v^i e_i \quad &\longleftrightarrow \quad v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p. \end{aligned}$$

Derivations at a Point

Remarks

1. From now on we will write a tangent vector as $v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$.
2. Although not geometric as the realization as arrows, the description of the tangent space in terms of derivations is more suitable for generalization to manifold (see Section 8).

Vector Fields

Definition (Vector Fields)

A vector field X on an open U of $\mathbb{R}^n$ is a map that assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$.

Remark

- As $T_p(\mathbb{R}^n)$ has basis $\{\frac{\partial}{\partial x^i}\big|_p\}$ for every p there are unique coefficients $a^j(p) \in \mathbb{R}$ such that

$$X_p = \sum a^j(p) \frac{\partial}{\partial x^i}\bigg|_p.$$

- We write $X = \sum a^j \frac{\partial}{\partial x^i}$, where the a^j are now functions on U.

Definition

A vector field $X = \sum a^j \frac{\partial}{\partial x^j}$ on U is C^∞ when the coefficient functions a^j are all C^∞ on U.

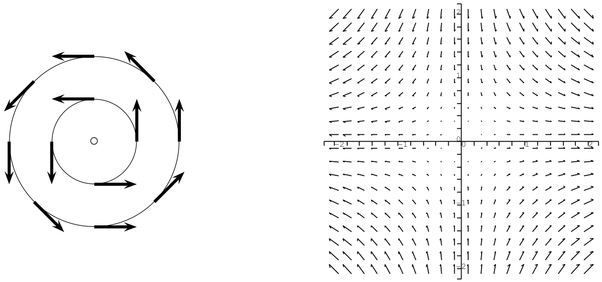
Example (Example 2.3)

$$X = \frac{-y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle,$$

$$Y = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \langle x, -y\rangle,$$

We may draw a vector at p as an arrow emanating from p (see next slide).

Vector Fields



(a) The vector field X on $\mathbb{R}^2 - \{\mathbf{0}\}$

(b) The vector field $\langle x, -y \rangle$ on $\mathbb{R}^2$

Figure 2.3 of Tu's book.

Reminder

1. The set of C^∞-functions on U is denoted by $C^\infty(U)$ or $\mathscr{F}(U)$.
2. This is an algebra over $\mathbb{R}$, and hence this is a ring.

Definition

$\mathscr{X}(u)$ is the set of all C^∞ vector fields on U.

Definition

We multiply a vector field X by a function f as follows:

$$(fX)_p := f(p)X_p, \qquad p \in U.$$

Remarks

1. If $X = \sum a^j \frac{\partial}{\partial x^j}$, then $fX = \sum (fa^j) \frac{\partial}{\partial x^j}$.
2. If $f \in C^\infty(U)$ and $X \in \mathscr{X}(U)$, then $fX \in \mathscr{X}(U)$.

Definition (Module over a Ring)

Let R be a commutative ring with identity 1. An R-module A is an Abelian group equipped with a scalar multiplication $R \times A \ni (r, a) \to ra \in A$ satisfying the following properties:

(i) Associativity: $(rs)a = r(sa)$ for all $r, s \in R$ and $a \in A$.

(ii) Identity: $1a = a$ for all $a \in A$.

(iii) Distributivity: $(r + s)a = ra + sa$ and $r(a + b) = ra + rb$.

Remark

When R is a field, an R-module is just a vector space over R.

Definition (R-Module Homomorphism)

Given R-modules A and A', an R-module homomorphism is an additive map $f : A \to A'$ that is compatible with scalar multiplication. That is,

$$f(a+b) = f(a) + f(b),$$
$$f(ra) = rf(a).$$

Fact

The set of C^∞ vector fields $\mathscr{X}(U)$ is a $C^\infty(U)$-module.

Vector Fields as Derivations

Definition

Given a C^∞ vector field X and a C^∞ function f on U, we define a function Xf by

$$(Xf)(p) = X_p f \qquad \text{for all } p \in U.$$

Equivalently, if we write $X = \sum a^i \frac{\partial}{\partial x^i}$, then

$$(Xf)(p) = \sum a^i(p) \frac{\partial f}{\partial x^i}(p).$$

Facts

1. The above formula shows that Xf is a C^∞ function.
2. We thus see that X defines a linear map,

$$X : C^\infty(U) \longrightarrow C^\infty(U).$$

Vector Fields as Derivations

Proposition (Proposition 2.6; Leibniz's Rule for Vector Fields)

Let X be a C^∞ vector field. Then it satisfies Leibniz's Rule,

$$X(fg) = fX(g) + (Xf)g \qquad \text{for all } f, g \in C^\infty(U).$$

Definition (Derivation of an Algebra)

1. If A is an algebra over a field $\mathbb{K}$, a derivation of A is any $\mathbb{K}$-linear map $D : A \to A$ that satisfies Leibniz's Rule.
2. The set of derivations of A is denoted by $\mathrm{Der}(A)$.

Fact

$\mathrm{Der}(A)$ is a vector space over $\mathbb{K}$.

Vector Fields as Derivations

Facts

1. We thus get a linear map,

$$\begin{array}{rcl} \varphi : \mathscr{X}(U) & \longrightarrow & \operatorname{Der}\left(C^\infty(U)\right), \\ X & \longrightarrow & \left(f \longrightarrow Xf\right). \end{array}$$

2. This map can be shown to a linear isomorphism.

Remark

Showing the injectivity of φ is not difficult, but the surjectivity requires some work (see Problem 19.12).

Consequence

In the same way we can identify tangent vectors at p and derivations at p, we may identify C^∞ vector fields on U with derivations of $C^\infty(U)$.

Vector Fields as Derivations

Remark

- A derivation at p is NOT a derivation of the algebra C_p^∞.
- A derivation at p is a linear map from C_p^∞ to $\mathbb{R}$, while a derivation of C_p^∞ is a linear map from C_p^∞ to itself.