Chap. 9 : K-Theory

1. Serre-Swan Theorem:

We saw duality: Spaces $\longleftrightarrow$ Algebras

We also have a duality: Vector bundles $\longleftrightarrow$ Finitely generated projective modules.

Def. Let A be an algebra (over $\mathbb{C}$). Then a finitely generated projective module is a right module $\mathcal{E}$ s.t.

$$\mathcal{E} \simeq p A^N, \quad p = (p_{ij}) \in M_N(A), \; p^2 = p.$$

$$\{(\textstyle\sum p_{ij} a_j)_{1 \leq i \leq N} \,;\; a_j \in A\} \subset A^N$$

Rmk: $p^2 = p \Rightarrow \begin{cases} p + (1-p) = 1 & \Rightarrow A^N = pA^N \oplus (1-p)A^N \\ p(1-p) = (p-1)p & \Rightarrow pA^N \text{ direct summand of } A^N. \end{cases}$

Let X be a compact Hausdorff topological space.

$A := C(X) =$ alg. of cont. functions on X

$\begin{array}{c} E \\ \downarrow \\ X \end{array}$ vector bundle $\rightsquigarrow$ $\mathcal{E} = C(X, E) =$ space of continuous sections of E

$\left.\begin{array}{l} f \in C(X) \\ s \in C(X,E) \end{array}\right\} \Rightarrow f(x)s(x)$ cont. section of E

$\rightsquigarrow$ $C(X, E)$ module over $C(X)$.

Thm.: (Serre-Swan) 1. $C(X, E)$ is a finitely generated projective module over $C(X)$.

2. If $\mathcal{E}$ is a finitely gen. proj. module over $C(X)$, then $\exists$ a vector bundle E s.t.

$$\mathcal{E} \simeq C(X, E).$$

2. K-Theory of a Space:

$X =$ compact Haus. top. space $\quad \mathcal{V}(X) = \{$finite-rank vector bundles over $X\}$

Def.: $E_1, E_2 \in \mathcal{V}(X)$ are stably equivalent and we write $E_1 \sim E_2$, when $\exists F \in \mathcal{V}(X)$ s.t.

$$E_1 \oplus F \simeq E_2 \oplus F$$

If $E_1 \sim E_2$ and $E_1' \sim E_2'$, then $E_1 \oplus E_1' \sim E_2 \oplus E_2'$. Therefore we get an addition on $\mathcal{V}(X)/_\sim$

$$\mathcal{V}(X)/_\sim = \{\underbrace{[E]}_{\text{stable equivalence class of } E};\; E \in \mathcal{V}(X)\}$$

$$[E_1] \oplus [E_2] := [E_1 \oplus E_2]$$

$\rightsquigarrow$ $\mathcal{V}(X)/_\sim$ is an Abelian monoid

Def: $K^0(X)$ is the Abelian group of formal differences
$$[E_1]-[E_2], \quad E_1, E_2 \in \mathcal{V}(X).$$

Rmk: Formal meaning of the definition: on $(\mathcal{V}(X)/_\sim)\times(\mathcal{V}(X)/_\sim)$ consider the equivalence relation
$$([E_1],[E_2]) \approx ([E_1'],[E_2']) \overset{\text{def}}{\Longleftrightarrow} [E_1]+[E_2'] = [E_1']+[E_2]$$
Then
$$K^0(X) = \left[(\mathcal{V}(X)/_\sim)\times(\mathcal{V}(X)/_\sim)\right]/\approx$$

Rmk: IN Abelian monoid and $\mathbb{Z} \simeq \{p-q;\ p,q \in \mathbb{N}\}$.

Functoriality: Let G be an Abelian group and assume there is an additive map
$$\varphi: \mathcal{V}(X)/_\sim \longrightarrow G.$$
Then φ uniquely extends to a group homomorphism
$$\varphi: K^0(X) \longrightarrow G.$$
$$\varphi([E_1]-[E_2]) := \varphi([E_1]) - \varphi([E_2]) \quad \forall E_1, E_2 \in \mathcal{V}(X).$$

Chern Character:

$M = C^\infty$ (compact) manifold

$E = C^\infty$-vector bundle

Chern character of E: $\mathrm{Ch}(E) = [\mathrm{Tr}\, e^{-F^\nabla}] \in H^{ev}(M,\mathbb{C})$

F^∇ = curvature connection ∇ on E.

$\mathrm{Ch}(E)$ does not depend on ∇

$$\mathrm{Ch}(E_1\oplus E_2) = \mathrm{Ch}(E_1) + \mathrm{Ch}(E_2)$$

$\rightsquigarrow$ We get additive map $\mathrm{Ch}: K^0(M) \longrightarrow \mathbb{Z}$.

Thm. (Atiyah-Hirzebruch) We get an isom.
$$\mathrm{Ch}: K^0(M)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\sim} H^{ev}(M,\mathbb{C}).$$

Ex: $X=\{pt\}$ $\quad \mathcal{V}(\{pt\}) = \{$ finite-dim. vector spaces over $\mathbb{C}\} =: \mathrm{Vect}(\mathbb{C})$

$\dim: \mathrm{Vect}(\mathbb{C}) \longrightarrow \mathbb{Z}$ $\qquad \dim(E_1\oplus E_2) = \dim E_1 + \dim E_2$

$E \longrightarrow \dim E$ $\qquad E_1 \simeq E_2 \Longleftrightarrow \dim E_1 = \dim E_2$ and $E_1\oplus F \simeq E_2 \oplus F \Rightarrow \dim E_1 = \dim E_2$

We get an isomorphism $\dim: K^0(\{pt\}) \xrightarrow{\sim} \mathbb{Z}$. $\quad$ & $E_1 \sim E_2 \Longleftrightarrow \dim E_1 = \dim E_2$.

$\dim([E_1]-[E_2]) = \dim E_1 - \dim E_2$ (virtual dimension).

3. K-Theory of an Algebra:

A = algebra over $\mathbb{C}$, $\quad$ A unital

$\mathcal{M}(A) = \{$ finitely generated projective modules over $A\}$

Def.: $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{M}(A)$ are <u>stably equivalent</u> when $\exists \mathcal{F} \in \mathcal{M}(A)$ s.t.
$$\mathcal{E}_1 \oplus \mathcal{F} \simeq \mathcal{E}_2 \oplus \mathcal{F}.$$

Def.: $K_0(A)$ is the Abelian group of formal differences
$$[\mathcal{E}_1] - [\mathcal{E}_2], \quad \mathcal{E}_1, \mathcal{E}_2 \in \mathcal{M}(A),$$
of ~~stable~~ stable equivalence classes of finitely gen. proj. modules over A.

Ex: $A = C(X)$, X compact space

Serre-Swan Thm. $\Rightarrow K_0(C(X)) \simeq K^0(X)$

In particular, $K_0(\underset{\text{algebra} = C(\{pt\})}{\mathbb{C}}) \simeq K^0(\{pt\}) \simeq \mathbb{Z}$.

$K_0(A)$ in terms of idempotents:

$\mathcal{E} \in \mathcal{M}(A) \iff \exists N \in \mathbb{N}\ \exists p \in M_N(A),\ p^2 = p,\ \mathcal{E} \simeq pA^N$.

(Murray-Von Neumann)

Def.: We say that two idempotents p_1 and p_2 of $M_N(A)$ are equivalent (and write $p_1 \sim p_2$) when $\exists u, v \in M_N(A)$ s.t. $uv = p_1$ and $vu = p_2$.

Lemma. Suppose that $\mathcal{E}_1 = p_1 A^N$ and $\mathcal{E}_2 = p_2 A^N$ where p_1 and p_2 are idempotents in $M_N(A)$. TFAE

(i) $\mathcal{E}_1 \simeq \mathcal{E}_2$ as modules over A.

(ii) $p_1 \sim p_2$.

Define

$$M_\infty(A) := \varinjlim M_N(A) := \{(a_{ij})_{i,j \in \mathbb{N}};\ \exists N \in \mathbb{N}\ a_{ij} = 0 \text{ for } i \geq N \text{ or } j \geq N\}.$$

Inclusion $M_N(A) \subset M_\infty(A)$ Then $M_\infty(A) = \bigcup M_N(A)$.

$$a \longrightarrow \left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right)$$

Identify $a \in M_N(A)$ w/ $b \in M_{N'}(A)$, $N' \geq N$, where $b = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right)$.

$M_\infty(A)$ is an algebra.

Whitney sum: $a \in M_N(A)$ $b \in M_{N'}(A)$
$$a \oplus b = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right) \in M_{N+N'}(A)$$

N.B.: $\mathcal{E}_1 = p_1 A^{N_1}$ $\mathcal{E}_2 = p_2 A^{N_2}$ p_1, p_2 idempotents.
$$\mathcal{E}_1 \oplus \mathcal{E}_2 \simeq \underbrace{(p_1 \oplus p_2)}_{\text{idempotent in } M_{N_1+N_2}(A)} A^{N_1+N_2}$$

Prop. TFAE

(i) $\mathcal{E}_1 \sim \mathcal{E}_2$

(ii) p_1 and p_2 are equivalent in $M_\infty(A)$.

Cor.: $K_0(A)$ is isomorphic to the Abelian group of formal differences

$$[p_1]-[p_2], \qquad p_1, p_2 \in M_\infty(A)$$

of equivalence classes of idempotents in $M_\infty(A)$.

Lemma: $K_0(A)=\{[p]-[I_N];\ p\in M_\infty(A),\ p^2=p\}$.

Proof: $[p_1]-[p_2]=[p_1]+[1-p_2]-([1-p_2]+[p_2]) \qquad p_j\in M_{N_j}(A)$

$$=[\underbrace{p_1\oplus(1-p_2)}_{p}]-[p_2\oplus(1-p_2)]$$

$$\overset{\shortparallel}{[I_{N_2}]} \quad \sim \quad N=2N_1$$

$$\begin{pmatrix}p_2&0\\0&1-p_2\end{pmatrix}\sim\begin{pmatrix}I_{N_2}&0\\0&0\end{pmatrix} \text{ using } u=\begin{pmatrix}p_2&0\\1-p_2&0\end{pmatrix}\ v=\begin{pmatrix}p_2&1-p_2\\0&0\end{pmatrix}$$

$$uv=\begin{pmatrix}p_2&0\\0&1-p_2\end{pmatrix} \quad vu=\begin{pmatrix}p_2+1-p_2&0\\0&0\end{pmatrix}$$

Functoriality: If $\varphi: A\to B$ morphism of algebras. Then we get morphisms

$\varphi: M_N(A)\to M_N(B)$

$\varphi: M_\infty(A)\to M_\infty(B)$ + $p_1\sim p_2 \Rightarrow \varphi(p_1)\sim\varphi(p_2)$

~~$\varphi: K_0(A) \to K_0(B)$~~ $\rightsquigarrow$ morphism $\varphi_*: K_0(A)\to K_0(B)$.

• **$K_0(A)$ when A is not unital:**

$A^+:=A\oplus\mathbb{C}$ unital algebra with product

$$(a\oplus\lambda)\cdot(b\oplus\mu)=(ab+\lambda b+\mu a,\ \lambda\mu)$$

. unit $(0,1)$. $A\simeq A\oplus\{0\}$ ideal of A^+.

Morphism $\varphi: A^+\to\mathbb{C}$

$(a,\lambda)\to\lambda$. $\rightsquigarrow$ morphism $\varphi: K_0(A^+)\to K_0(\mathbb{C})\simeq\mathbb{Z}$

Def: $K_0(A)=\ker\{\varphi: K_0(A^+)\to\mathbb{Z}\}$.

Prop:

$$K_0(A)=\{[p]-[I_N];\ p\in M_\infty(A^+),\ p^2=p \text{ and } p-I_N\in M_\infty(A)\}$$

Case: When A is unital we recover the previous defn of $K_0(A)$.

• **Topological Description of $K_0(A)$:**

Assume A is a unital Banach algebra.

Def: Let p, q be two idempotents in $M_N(A)$.

(1) p and q are similar, and we write $p\sim_s q$, when $\exists g\in GL_N(A)$ s.t. $q=g^{-1}pg$.

(2) p and q are homotopic, and we write $p\sim_h q$, when $\exists$ exists a continuous path $[0,1]\ni t\to p(t)\in M_N(A)$ s.t. $p(t)^2=p(t)$, $p(0)=p$ and $p(1)=q$.

Lemma: Let p, q be idempotents in $M_N(A)$. Then:
(i) $p \sim q$ in $M_N(A) \Rightarrow p \sim_s q$ in $M_{2N}(A)$.
(ii) $p \sim_s q$ in $M_N(A) \Rightarrow p \sim q$ in $M_N(A)$
(iii) $p \sim_h q$ in $M_N(A) \Rightarrow p \sim_s q$ in $M_N(A)$
(iv) $p \sim_s q$ in $M_N(A) \Rightarrow p \sim_h q$ in $M_{2N}(A)$.

Prop. Let p, q be idempotents in $M_\infty(A)$. Then
$$p \sim q \Leftrightarrow p \sim_s q \Leftrightarrow p \sim_h q$$
Consequence: $K_0(A)$ is isomorphic to the Abelian group of formal differences
$$[p]-[q],$$
of homotopy classes of idempotents in $M_\infty(A)$.

. $K_0(A)$ when A is a C^*-algebra:
Assume A is a unital C^*-algebra. $\leadsto$ $M_N(A)$ C^*-algebra $\forall N$ $(a_{ij})^* = (a_{ji}^*)$
$\|a\| = \sup_{\substack{\xi \in A^N \\ \|\xi\|=1}} \|a\xi\|$
Def.: A selfadjoint idempotent $p \in M_N(A)$ is called a projection.
Lemma. Let $p, q \in A$ be projections. Then:
(i) $p \sim q \Leftrightarrow \exists u \in A$ s.t. $u^*u = p$ and $uu^* = q$ (Murray-von Neumann equivalence)
(ii) $p \sim_s q \Leftrightarrow \exists u \in A$ unitary $u^*pu = q$ (unitary equivalence)
(iii) $p \sim_h q \Leftrightarrow p$ and q can be connected by a continuous path of projections (homotopy equivalence of proj.)
Prop. If A is a C^*-algebra, then $K_0(A)$ is isomorphic to the group of formal differences
$$[p]-[q]$$
of Murray-von Neumann / unitary / homotopy equivalence classes of projections in $M_\infty(A)$.
Let B be a subalgebra of A.
Def. B is C^∞-closed, or closed under the holomorphic functional calculus when
(i) If $1 \in B$: $\forall b \in B$ $\forall f \in \mathrm{Hol}(\mathrm{Sp}\, b)$ $f(b) \in B$
(ii) If $1 \notin B$: $\forall b \in B$ $\forall f \in \mathrm{Hol}(\mathrm{Sp}\, b)$ $f(0)=0$. $f(b) \in B$.
Rmk: If B is equipped with a Fréchet algebra topology finer than the topology induced by A, then
$$B \text{ is } C^\infty\text{-closed} \Leftrightarrow \forall b \in B \begin{cases} b \text{ invertible in } A \Rightarrow b^{-1} \in B \; (1 \in B) \\ 1+b \text{ ——} \Rightarrow (1+b)^{-1}-1 \in B \; (1 \notin B) \end{cases}$$

Ex: (1) M compact mfld $C^\infty(M)$ is C^∞-closed in $C^0(M)$
(2) $\mathcal{S}(\mathbb{R}^n)$ is C^∞-closed in $C_0(\mathbb{R}^n)$

Lemma: B C^∞-closed $\Rightarrow M_N(A)$ C^∞-closed $\forall N \in \mathbb{N}$.

Suppose that B is a dense subalgebra of A such that

(i) B is involutive: $b \in B \Rightarrow b^* \in B$

(ii) B is C^∞-closed.

The inclusion $\iota: B \hookrightarrow A$ gives rise to a morphism

$$\iota_*: K_0(B) \longrightarrow K_0(A).$$

Lemma: (i) Let $p, q \in M_N(B)$ be projections s.t. $p \sim q$ in $M_N(A)$. Then p and q are unitarily equivalent in $M_N(B)$.

(ii) Let $p \in M_N(A)$ be an idempotent. Then $\exists$ projection $q \in M_N(B)$ s.t. $p \sim_s q$ in $M_N^{inv}(A)$.

(i) implies that ι_* is one-to-one.

(ii) implies ____ is onto.

Prop. $K_0(A)$ and $K_0(B)$ are isomorphic to the Abelian group of formal differences

$$[p] - [q]$$

of unitarily equivalence classes of projections in $M_\infty(B)$.

Ex. $K_0(C^\infty(M)) \simeq K_0(C(M))$. ($M$ compact mfld)

Serre-Swan $K_0(C^0(M)) \simeq K^0(M)$ + $K_0(C^\infty(M)) \simeq K_0(M)^{C^\infty\text{-cont}}$

$\Rightarrow K_0(M) \simeq \{[E] - [F];\ E, F\ C^\infty\text{-vector bundles over } M\}$.

Chap. 10: Cyclic Cohomology

1. Pairing K-Theory / de Rham Homology:

M = compact C^∞-manifold

Chern character $Ch: K^0(M) \to H^{ev}(M, \mathbb{C})$

$$Ch([E_1]-[E_2]) = [\operatorname{Tr} e^{-F^{E_1}}] - [\operatorname{Tr} e^{-F^{E_2}}]$$

F^{E_i} = curvature some connection on E.

Def. $k = 0, 1, \ldots, n$ define $\mathcal{D}'_k(M) = C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M)'$. $\quad \Omega^k(M) := C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M)$

elements of $\mathcal{D}'_k(M)$ = k-dimensional currents.

Duality $C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M) \times \mathcal{D}'_k(M) \to \mathbb{C}$.

de Rham coboundary gives

$$d: C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M) \to C^\infty(M, \Lambda^{k+1}_{\mathbb{C}} T^*M)$$

By duality we get a boundary map

$$d^t: \mathcal{D}'_k(M) \to \mathcal{D}'_{k-1}(M)$$

$$\langle d^t C, \omega \rangle := \langle C, d\omega \rangle \quad \forall C \in \mathcal{D}'_k(M) \ \forall \omega \in C^\infty(M, \Lambda^{k-1}_{\mathbb{C}} T^*M).$$

$d^2 = 0 \Rightarrow (d^t)^2 = 0$.

The homology of the complex $(\mathcal{D}'_*(M), d^t)$ is called the de Rham homology of M and is denote $H_*(M)$.

We have a natural pairing:

$$H_k(M) \times H^k(M) \to \mathbb{C}$$

Composing it w/ the Chern character $Ch: K^0(M) \to H^{ev}(M)$ we get a pairing

$$H_{ev}(M) \times K^0(M) \to \mathbb{C}$$

$$\langle [C], [E] \rangle = \langle C, \operatorname{Tr} e^{-F^E} \rangle \qquad \forall C \in \mathcal{D}'_{ev}(M) = \bigoplus_{k\, ev} \mathcal{D}'_k(M)$$

$\forall$ vector bundle E over M w/ curvature F^E

Ex: Assume M oriented. Let $\omega \in \Omega^{n-k}(M)$. Then ω defines a k-dimensional current $\omega^\wedge$ by

$$\langle \omega^\wedge, \eta \rangle := \int_M \underbrace{\omega \wedge \eta}_{\in \Omega^n(M)} \qquad \forall \eta \in \Omega^k(M).$$

$\omega^\wedge$ = Poincaré dual of ω

2. Hochschild and Cyclic Cohomology:

A = algebra over $\mathbb{C}$ with unit.

For $n \in \mathbb{N}_0$ define

$$C^n(A) = \{(n+1)\text{-linear forms } \varphi : A^{n+1} \to \mathbb{C}\}$$

Hochschild coboundary operator

$$b : C^n(A) \to C^{n+1}(A)$$

$$b\varphi(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \dots, a^n)$$

Ex: $n = 0$ $C^0(A) = \{\text{linear forms } \varphi : A \to \mathbb{C}\}$.

$$b\varphi(a^0, a^1) = \varphi(a^0 a^1) - \varphi(a^1 a^0)$$

$$b\varphi = 0 \iff \varphi \text{ is a trace.}$$

Lemma: $b^2 = 0$.

Def.: The cohomology of the complex $(C^*(A), b)$ is called the Hochschild cohomology of A and is denoted $HH^*(A)$.

$$HH^n(A) = \frac{Z^n(A)}{B^n(A)} \qquad Z^n(A) = \{\varphi \in C^n(A),\ b\varphi = 0\}$$

$$B^n(A) = b\,C^{n-1}(A).$$

Def. (1) A cochain $\varphi \in C^n(A)$ is cyclic when

$$\varphi(a^1, \dots, a^n, a^0) = (-1)^n \varphi(a^0, a^1, \dots, a^n).$$

(2) We define

$$C_\lambda^n(A) = \{\varphi \in C^n(A);\ \varphi \text{ cyclic}\}.$$

Define $A : C^n(A) \to C^n(A)$ by

$$(A\varphi)(a^0, \dots, a^n) = \sum (-1)^{nj} \varphi(a^j, \dots, a^n, a^0, \dots, a^{j-1})$$

Then:

$$\varphi \in C_\lambda^n(A) \iff \varphi = \frac{1}{n+1} A\varphi.$$

Define $b' : C^n(A) \to C^{n+1}(A)$ by

$$b'\varphi(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^n)$$

Lemma: $bA = Ab'$

Consequence: $b\,C_\lambda(A) = b\,A C^n(A) = A b' C^n(A) \subset C_\lambda^{n+1}(A)$

$\leadsto$ The cyclic complex is a sub-complex of the Hochschild complex.

Def. The cohomology of the complex $(C_\lambda^*(A), b)$ is called the cyclic cohomology of A and is denoted

$$HC^*(A) = \frac{Z_\lambda^n(A)}{B_\lambda(A)} \qquad Z_\lambda^n(A) = \{\varphi \in C_\lambda^n(A);\ b\varphi = 0\}$$

$$B_\lambda^n(A) = b\,C_\lambda^{n-1}(A).$$

3. Periodic Cyclic Cohomology:

The operator $B: C^{n+1}(A) \longrightarrow C^n(A)$ is defined by

$$B = AB_0, \quad ?$$

where $B_0: C^{n+1}(A) \to C^n(A)$ is given by

$$B_0\varphi(a^0,\dots,a^n) = \varphi(1,a^0,\dots,a^n) - (-1)^{n+1}\varphi(a^0,\dots,a^n,1)$$

Rmk: $C_\lambda(A) \subset \ker B_0$ as $B_0 A = 0$.

Lemma: (i) $B^2 = 0$,

(ii) $bB + Bb = 0$.

Define

$$C^{ev}(A) = \bigoplus_{n \text{ even}} C^n(A) \quad \text{and} \quad C^{odd}(A) = \bigoplus_{n \text{ odd}} C^n(A)$$

Then we have a short complex

$$C^{ev}(A) \underset{\longleftarrow}{\xrightarrow{b+B}} C^{odd}(A), \qquad (b+B)^2 = 0.$$

Def. The cohomology of the complex $(C^{ev/odd}(A), b+B)$ is called the periodic cyclic cohomology of A. Its cohomology groups are denoted

$$HC^{ev}(A) = \frac{Z^{ev}(A)}{B^{ev}(A)} \quad \text{and} \quad HC^{odd}(A) = \frac{Z^{odd}(A)}{B^{odd}(A)}.$$

$C^{ev/odd}(A) = \{\varphi = (\varphi_n)_{n \text{ ev/odd}};\ \varphi_n \in C^n(A),\ \varphi_n = 0 \text{ for } n \text{ large enough}\}$.

Let $\varphi = (\varphi_n)_{n \text{ even}} \in C^{ev}(A)$. Then

$$\varphi \in Z^{ev}(A) \iff (b\varphi_n) + (B\varphi_n) = 0 \iff \underset{C^{n+1}}{\underset{\cap}{b\varphi_n}} + \underset{C^{n+1}}{\underset{\cap}{B\varphi_{n+2}}} = 0 \quad \forall n$$

$$\varphi \in B^{ev}(A) \iff \exists \psi = (\psi_n)_{n \text{ odd}} \in C^{odd}(A) \text{ s.t. } \varphi = (b\psi_n) + (B\psi_n)$$

$$\iff \exists \psi = (\psi_n) \text{ ——— } \varphi_n = \underset{C^n}{\underset{\cap}{b\psi_{n-1}}} + \underset{C^n}{\underset{\cap}{B\psi_{n+1}}} \quad \forall n.$$

4. Cyclic Cohomology of $A = C^\infty(M)$:

$A = C^\infty(M)$, M closed manifold of dim. n.

$C^k(A) = \{\underline{\text{continuous}}\ (n+1)\text{-linear forms } \varphi: C^\infty(M)^{k+1} \to \mathbb{C}\}$.

$C \in \mathcal{D}'_k(M) \to \varphi_C \in C^k(A)$ defined by

$$\varphi_C(f^0, f^1, \dots, f^k) = \langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle.$$

Thm. (Connes, Hochschild-Kostant-Rosenberg) The linear map $C \to \varphi_C$ gives rise to an isomorph.

$$\mathcal{D}'_k(M) \simeq HH^k(A)$$

↑ Hochschild coham. of $\underline{\text{continuous}}$ cochains.

Therefore:

(i) $\forall \varphi \in C^{\ell}(A)\ \exists C \in \mathcal{D}'_{\ell}(M)\ \exists \psi \in C^{\ell-1}(A)$ s.t. $\varphi = \varphi_C + b\psi$.

(ii) $\forall C \in \mathcal{D}'_{\ell}(M)$ $C = 0 \iff \exists \psi \in C^{\ell-1}(A)$ s.t. $\varphi_C = b\psi$.

Let $C \in \mathcal{D}'_{\ell}(M)$. Then

$$\begin{aligned} B_0\varphi_C(f^0, f^1, \ldots, f^{\ell-1}) &= \varphi_C(1, f^0, \ldots, f^{\ell-1}) - (-1)^{\ell}\varphi_C(f^0, \ldots, f^{\ell-1}, 1) \\ &= \langle C, 1 df^0 \wedge \cdots \wedge df^{\ell-1}\rangle - (-1)^{\ell} \underbrace{\langle C, f^0 df^1 \wedge \cdots \wedge d(1)\rangle}_{=0} \\ &= \langle C, d(f^0 df^1 \wedge \cdots \wedge df^{\ell-1})\rangle \\ &= \langle d^t C, f^0 df^1 \wedge \cdots \wedge df^{\ell-1}\rangle \\ &= \varphi_{d^tC}(f^0, \ldots, f^{\ell-1}) \quad \leadsto \quad B_0\varphi_C = \varphi_{d^tC}. \end{aligned}$$

Moreover:

$$\begin{aligned} \varphi_{d^tC}(f^0, \ldots, f^{\ell-1}) = \langle C, df^0 \wedge \cdots \wedge df^{\ell-1}\rangle &= (-1)^{\ell-1}\langle C, df^1 \wedge \cdots \wedge df^{\ell-1} \wedge df^0\rangle \\ &= (-1)^{\ell-1}\varphi_{d^tC}(f^1, \ldots, f^{\ell-1}, f^0) \end{aligned}$$

$\leadsto \varphi_{d^tC} \in C_\lambda^{\ell-1}(A) \leadsto A\varphi_{d^tC} = \ell\, \varphi_{d^tC}$.

Therefore:

$$B\varphi_C = AB_0\varphi_C = A\varphi_{d^tC} = \ell\, \varphi_{d^tC}.$$

For $C \in \mathcal{D}'_{ev/odd}(M) := \bigoplus_{\ell \text{ even/odd}} \mathcal{D}'_{\ell}(M)$ $(= (C_\ell)_{\text{even/odd}})$ define $\varphi_C \in C^{ev/odd}(A)$ by

$$\varphi_C = \left(\frac{1}{\ell!}\varphi_{C_\ell}\right)_{\ell \text{ even/odd}}.$$

Then

$$(b+B)\varphi_C = B\varphi_C = \left(\frac{1}{\ell!}B\varphi_{C_\ell}\right) = \left(\frac{\ell}{\ell!}\varphi_{d^tC_\ell}\right) = \left(\frac{1}{(\ell-1)!}\varphi_{d^tC_\ell}\right)$$

$$\leadsto (b+B)\varphi_C = \varphi_{d^tC}.$$

<u>Thm. (Connes)</u>: The linear map $\mathcal{D}'_{ev/odd} \ni C \longmapsto \varphi_C \in C^{ev/odd}(A)$ gives rise to isomorphisms

$$H_{ev/odd}(M) \simeq HC^{ev/odd}(A)$$

periodic cyclic cohomology of *continuous* cochains.

<u>5. Pairing w/ K-theory ($A = C^\infty(M)$):</u>

We have a pairing $H_{ev}(M) \times K^0(M) \longrightarrow \mathbb{C}$

$$\langle [C], [E]\rangle = \langle C, \operatorname{Tr} e^{-F^E}\rangle.$$

$\leadsto$ yields a pairing $HC^{ev}(A) \times K_0(A) \longrightarrow \mathbb{C}$, $A = C^\infty(M)$.

Suppose $E = \operatorname{ran} p$, $p \in M_N(A)$, projection.

$\rightsquigarrow C^\infty(M,E) \simeq pA^N = \{\xi \in C^\infty(M,\mathbb{C}^N);\ p\xi = \xi\} \subset C^\infty(M,\mathbb{C}^N)$

The Grassmannian connection of E is defined by

$$\begin{tikzcd} C^\infty(M,E) \arrow[r, hook] \arrow[drr, "\nabla_0"'] & C^\infty(M,\mathbb{C}^N) \arrow[r, "d"] & C^\infty(M, T^*M\otimes\mathbb{C}^N) \arrow[d, "1\otimes p"] \\ & & C^\infty(M, T^*M\otimes E) \end{tikzcd}$$

<u>Lemma</u>: Let F^{∇_0} be the curvature of ∇_0. Then:

(1) $F^{\nabla_0} = p(dp)^2 = p(dp)p$.

(2) $\operatorname{Tr} e^{-F^{\nabla_0}} = \sum_{\ell\geq 0} \frac{(-1)^\ell}{\ell!} \operatorname{Tr}[p(dp)^{2\ell}]$.

• $\varphi \in C^\ell(A)$, Tr = trace on $M_N(\mathbb{C})$ $\Big\}$ $\operatorname{Tr}\#\varphi \in C^\ell(M_\ell(A))$

$\operatorname{Tr}\#\varphi(\mu^0\otimes a^0,\dots,\mu^\ell\otimes a^\ell) := \operatorname{Tr}[\mu^0\cdots\mu^\ell]\,\varphi(a^0,\dots,a^\ell)$.

$\varphi = \varphi_C$, $C \in \mathcal{D}'_\ell$: $\operatorname{Tr}\#\varphi_C(\mu^0\otimes f^0,\dots,\mu^\ell\otimes f^\ell) = \operatorname{Tr}[\mu^0\cdots\mu^\ell]\,\langle C, f^0 df^1\wedge\dots\wedge df^\ell\rangle$

$= \langle C, \operatorname{Tr}[\mu^0\cdots\mu^\ell]\, f^0 df^1\wedge\dots\wedge df^\ell\rangle$

$= \langle C, \operatorname{Tr}[\mu^0\otimes f^0\, d(\mu^1\otimes f^1)\cdots d(\mu^\ell\otimes f^\ell)]\rangle$

$\operatorname{Tr}\#\varphi_C(a^0,\dots,a^\ell) = \langle C, \operatorname{Tr}[a^0 da^1\cdots da^\ell]\rangle \quad \forall a^j \in M_N(C^\infty(M))$.

Therefore for $C = (C_\ell)_{\ell\ \text{even}} = (C_{2\ell})_{\ell\geq 0} \in \mathcal{D}'_{ev}(M)$

$$\langle C, \operatorname{Tr} e^{-F^{\nabla_0}}\rangle = \sum_{\ell\geq 0}\frac{(-1)^\ell}{\ell!}\langle C_{2\ell}, \operatorname{Tr}[p(dp)^{2\ell}]\rangle$$

$$= \sum_{\ell\geq 0}\frac{(-1)^\ell}{\ell!}\operatorname{Tr}\#\varphi_{C_{2\ell}}(p,p,\dots,p) \qquad \varphi_C = \left(\frac{1}{(2\ell)!}\varphi_{C_{2\ell}}\right)$$

$$\langle [C],[E]\rangle = \sum_{\ell\geq 0}(-1)^\ell\frac{(2\ell)!}{\ell!}\operatorname{Tr}\#(\varphi_C)_{2\ell}(p,p,\dots,p)$$

<u>6. Pairing w/ K-theory (General Case)</u>:

For $\varphi \in C_\lambda^{2\ell}(A)$ and $e \in M_N(A)$, $e^2 = e$, we define

$$\langle\varphi, e\rangle := \frac{1}{\ell!}\operatorname{Tr}\#\varphi(e,\dots,e)$$

<u>Lemma</u>: (i) If $\varphi = b\psi$, $\psi \in C_\lambda^{2\ell-1}(A)$, then $\langle\varphi, e\rangle = 0$

(ii) $\langle\varphi, geg^{-1}\rangle = \langle\varphi, e\rangle \ \forall g \in GL_N(A)$

Thm. (1) For $\varphi \in Z_\lambda^{2\ell}(A)$ and $e \in M_N(A)$, $e^2=e$, the pairing $\langle\varphi, e\rangle$ depends only on the class of φ in $HC^{2\ell}(A)$ and e in $K_0(A)$.

(2) This gives rise to a ~~bilinear~~ pairing

$$\langle\cdot,\cdot\rangle : HC^{2\ell}(A)\times K_0(A) \longrightarrow \mathbb{C}$$

Def. (1) $\varphi \in C^n(A)$ is normalized when

$$\varphi(a^0, a^1, \ldots, a^n) = 0 \text{ when } a^j = 1 \text{ for some } j\geq 1.$$

Rmk: φ normalized $\Rightarrow$ $B_0\varphi \in C^{n-1}(A)$, $B_0\varphi(a^0,\ldots,a^{n-1}) = \varphi(1,a^0,\ldots,a^{n-1}) - (-1)^n\varphi(a^0,\ldots,a^{n-1},1) = \varphi(1,a^0,\ldots,a^{n-1})$

(2) $\varphi = (\varphi_n)_{n \text{ ev/odd}} \in C^{ev/odd}(A)$ is normalized when each component φ_n is normalized in the sense of (1).

Define $C_0^{ev/odd}(A) = \{\varphi \in C^{ev/odd}(A);\ \varphi \text{ normalized}\}$.

Lemma. $(C_0^{ev/odd}(A), b+B)$ is a subcomplex of $(C^{ev/odd}(A), b+B)$ w/ same cohomology.

For $\varphi = (\varphi_{2\ell}) \in C_0^{ev/odd}(A)$ normalized and $e \in M_N(A)$, $e^2=e$, we define:

$$\langle\langle\varphi, e\rangle\rangle = \mathrm{Tr}\#\varphi_0(e) + \sum_{\ell\geq 1}(-1)^\ell\frac{(2\ell)!}{\ell!}\mathrm{Tr}\#\varphi_{2\ell}(e-\tfrac{1}{2}, e, \ldots, e)$$

Rmk: $\varphi_{2\ell}$ normalized $\Rightarrow$ $B_0\varphi(e,\ldots,e) = \varphi(1,e,\ldots,e)$

$$\rightsquigarrow \langle\langle\varphi,e\rangle\rangle = \mathrm{Tr}\#\varphi_0(e) + \sum_{\ell\geq 1}(-1)^\ell\frac{(2\ell)!}{\ell!}\left\{\mathrm{Tr}\#\varphi_{2\ell}(e,\ldots,e) - \frac{1}{2}\mathrm{Tr}\#B_0\varphi(e,\ldots,e)\right\}$$

Lemma: (1) If $\varphi = (b+B)\psi$ w/ $\psi \in C^{odd}(A)$ normalized, then $\langle\langle\varphi,e\rangle\rangle = 0$

(2) $\langle\langle\varphi, g^{-1}eg\rangle\rangle = \langle\langle\varphi,e\rangle\rangle$ $\forall g \in GL_N(A)$

Consequence: $\langle\langle\varphi,e\rangle\rangle$ depends only on the class of φ in $HC^{ev}(A)$ and e in $K_0(A)$.

We thus get a pairing:

$$\langle\langle\cdot,\cdot\rangle\rangle : HC^{ev}(A)\times K_0(A) \longrightarrow \mathbb{C}.$$

Ex: $A = C^\infty(M)$ E = range e, $e \in M_N(A)$, $e^2=e$. $C \in \mathcal{D}'_{ev}(M)$, $\partial^t C = 0$

$\rightsquigarrow \varphi_C$ normalized and $\varphi_C(1, f^1, \ldots, f^{2\ell}) = 0$.

$$\langle\langle\varphi_C, e\rangle\rangle = \langle C, \mathrm{Tr}\, e^{-F^{\nabla_0}}\rangle$$

$$= \langle [C], [E]\rangle \qquad F^{\nabla_0} = \text{curvature of Grassmann connection}$$

Chapter II: The Local Index Formula in NCG

1. Spectral Triples:

Spectral triple = NC substitute for a manifold

An (even) spectral triple is a triple $(A, \mathcal{H}, D)$, where:

- $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a $\mathbb{Z}_2$-graded Hilbert space (we shall denote by $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the grading operator so that $\gamma^2 = 1$ and $\gamma^* = \gamma$).
- A is a unital $*$-algebra together w/ a $*$-representation $\pi : A \to \mathcal{L}(\mathcal{H})$ (i.e., $\pi(a^*) = \pi(a)^*$) so that

$$\gamma \pi(a) = \pi(a) \gamma \quad \forall a \in A \quad \left(\text{i.e., } \pi(a) = \begin{pmatrix} \pi(a)^+ & 0 \\ 0 & \pi(a)^- \end{pmatrix} \ \forall a \in A \right).$$

(Usually the representation π is dropped from the notation, i.e., we identify A and $\pi(A)$)

- D is a (densely defined) selfadjoint unbounded operator on $\mathcal{H}$ so that
  - $\gamma D = -D\gamma$, i.e., $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, $D^\pm : \operatorname{dom} D \cap \mathcal{H}^\pm \to \mathcal{H}^\mp$.
  - $[D, a] \in \mathcal{L}(\mathcal{H})$ $\forall a \in A$, i.e., $a(\operatorname{dom} D) \subset \operatorname{dom} D$ and $aD - Da$ is bounded $\forall a \in A$.
  - D has compact resolvent, i.e., $(D+i)^{-1}$ is a compact operator.

In addition, we shall assume:

- the algebra A is closed under holomorphic functional calculus.

<u>Rmk.</u> The last assumption implies that

$$K_0(A) = K_0(\bar{A}) = \left\{ [e_1] - [e_2];\ e_j \in M_\infty(A),\ e_j^* = e_j^2 = e_j \right\}$$

<u>Example (Dirac Spectral Triple):</u>

M = cpct spin Riem. oriented mfld of even dim.

$\$ = \$^+ \oplus \$^-$ spinor bundle

$\not{D} : C^\infty(M, \$) \to C^\infty(M, \$)$ Dirac op. of M. $\quad \$ = \$^+ \oplus \$^-$

Then $(C^\infty(M), L^2(M, \$), \not{D})$ is a spectral triple, where:

- $C^\infty(M)$ is represented in $L^2(M, \$)$ by multiplication operators.
- $L^2(M, \$)$ is equipped w/ the $\mathbb{Z}_2$-grading $L^2(M, \$) = L^2(M, \$^+) \oplus L^2(M, \$^-)$.

- $D\!\!\!/$ is regarded as an unbounded (selfadjoint) op. on $L^2(M, \$)$ w/ domain the Sobolev space $L^2_1(M, \$)$.

2. The Index Map of a Dirac Spectral Triple:

M = cpct Riem. spin Riem. oriented mfld. of even dim

$\$ = \$^+ \oplus \$^-$ = spinor bundle w/ spin connection $\nabla^{\$}$

$D\!\!\!/ : C^\infty(M, \$) \longrightarrow C^\infty(M, \$)$ = Dirac op.

E = Hermitian vector bundle w/ Hermitian connection ∇^E

Twisted Dirac op.

$$D\!\!\!/_E : C^\infty(M, \$\otimes E) \xrightarrow{\nabla^{\$}\otimes 1 + 1\otimes\nabla^E} C^\infty(M, \$\otimes T^*M\otimes E) \xrightarrow{c\otimes 1} C^\infty(M, \$\otimes E)$$

$$\sigma\otimes\omega\otimes\xi \longrightarrow (c(\omega)\sigma)\otimes\xi$$

$$D\!\!\!/ : C^\infty(M, \$) \xrightarrow{\nabla^{\$}\otimes 1} C^\infty(M, \$\otimes T^*M) \xrightarrow{c} C^\infty(M, \$)$$

$$D\!\!\!/_E = D\!\!\!/\otimes 1 + (c\otimes 1)(1\otimes\nabla^E)$$

subbundle of the trivial bundle $M\times\mathbb{C}^N$

Assume that: - $E = \operatorname{ran}(p)$, $p\in M_N(C^\infty(M)) = C^\infty(M, M_N(\mathbb{C}))$, $p^2 = p^* = p$.

- $\nabla^E = \nabla^0 = (1\otimes p)(d\otimes 1)$ = Grassmanian connection

Rmk: ∇^0 is a Hermitian connection w/ respect to the Hermitian metric on E induced by that of the trivial bundle $M\times\mathbb{C}^q$:

$$\langle\xi,\eta\rangle_{M\times\mathbb{C}^q}(x) = \langle\xi(x),\eta(x)\rangle_{\mathbb{C}^q}\quad\forall\xi,\eta\in C^\infty(M,\mathbb{C}^q)$$

Let $\xi = (\xi_j)\in C^\infty(M, E) = \{\xi\in C^\infty(M,\mathbb{C}^N);\ p\xi = \xi\} \simeq p\,C^\infty(M)^N$

$p = (p_{ij})$

$$\nabla^0\xi = p\,d\xi = \Big(\sum_j p_{ij}\,\xi_j\Big)_{1\leq i\leq N}$$

Let $\sigma\in C^\infty(M, \$)$. Then

$$(\nabla^{\$}\otimes 1 + 1\otimes\nabla^0)(\sigma\otimes\xi) = (\nabla^{\$}\sigma)\otimes\xi + \sigma\otimes p\,d\xi$$

$$(\nabla^{\$}\otimes 1 + 1\otimes\nabla^0)(\sigma\otimes\xi)_i = \xi_i\nabla^{\$}\sigma + \sum_j p_{ij}\,\sigma\otimes d\xi_j$$

Thus,

$$D\!\!\!/_E(\sigma\otimes\xi)_i = c\otimes 1\Big(\xi_i\nabla^{\$}\sigma + \sum_j p_{ij}\,\sigma\otimes d\xi_j\Big)$$

$$= \xi_i\,c(\nabla^{\$}\sigma) + \sum_j p_{ij}\,c(d\xi_j)\sigma$$

We have

- $c(\nabla^{\$}\sigma) = D\!\!\!/\sigma$

- $c(d\xi_j)\sigma = \sigma_1(D\!\!\!/)(d\xi_j)\sigma = [D, \xi_j]\sigma = D\!\!\!/(\xi_j\sigma) - \xi_j D\!\!\!/\sigma$

Therefore

$$\not{D}_E(\sigma\otimes\xi)_i = \xi_i \not{D}\sigma + \sum_j \left\{ p_{ij}\, \not{D}(\xi_j\sigma) - p_{ij}\xi_j \not{D}\sigma \right\}$$

$$= \xi_i \not{D}\sigma + \sum_j p_{ij}\, \not{D}(\xi_j\sigma) - \Big(\sum_j p_{ij}\xi_j\Big)\not{D}\sigma$$

$$(p\xi)_i = \xi_i$$

$$= \cancel{\xi_i \not{D}\sigma} + \big(p\,(\not{D}\otimes I_N)(\sigma\otimes\xi)\big)_i - \cancel{\xi_i \not{D}\sigma}.$$

Thus,

$$\not{D}_E = p\,(\not{D}\otimes I_N) : \underbrace{p\,C^\infty(M,S)^N}_{\cong\, C^\infty(M,\, S\otimes E)} \longrightarrow \underbrace{p\,C^\infty(M,S)^N}_{\cong\, C^\infty(M,\, S\otimes E)}$$

3. The Index Map of a Spectral Triple:

Let $(A, \mathcal{H}, D)$ be a spectral triple. Notice that the fact that $(D+i)^{-1}$ is compact implies that the spectrum of D consists of isolated (real) eigenvalues w/ finite multiplicities.

In the sequel we denote by Π_0 the orthogonal projection onto $\ker D$ and by D^{-1} its partial inverse. So D^{-1} is the bounded operator of $\mathcal{H}$ that vanishes on $\ker D$ and inverts D on $(\ker D)^\perp = \operatorname{im} D$. Equivalently, $D^{-1} = (D+\Pi_0)^{-1} - \Pi_0$. Notice that $\mathcal{H}'$

$$DD^{-1} = 1-\Pi_0 \quad\text{and}\quad D^{-1}D = 1-\Pi_0 \text{ on } \operatorname{dom} D.$$

In addition, we denote by $\mathcal{H}_1$ the Hilbert space consisting of $\operatorname{dom} D$ equipped w/ the inner product and norm,

$$\langle \xi, \eta\rangle_1 := \langle \xi, \eta\rangle + \langle D\xi, D\eta\rangle \quad\text{and}\quad \|\xi\|_1 := \sqrt{\|\xi\|^2 + \|D\xi\|^2}.$$

Notice that D is a continuous operator from $\mathcal{H}_1$ to $\mathcal{H}$.

Lemma 11.1: (i) The inclusion $\iota : \mathcal{H}_1 \hookrightarrow \mathcal{H}$ is compact.

(ii) D^{-1} is a compact op. of $\mathcal{H}$ and is continuous from $\mathcal{H}$ to $\mathcal{H}_1$.

Proof: Upon writing $\iota = \underbrace{(D+i)^{-1}}_{\mathcal{K}(\mathcal{H})} \cdot \underbrace{(D+i)}_{\mathcal{L}(\mathcal{H}_1,\mathcal{H})}$ and $D^{-1} = D^{-1}(D+i)\cdot(D+i)^{-1} = \underbrace{(1-\Pi_0+iD^{-1})}_{\mathcal{L}(\mathcal{H})} \cdot \underbrace{(D+i)}_{\mathcal{B}(\mathcal{H})}$ we see that ι is compact and D^{-1} is a compact op. of $\mathcal{H}$.

Let $\xi \in \mathcal{H}$. Then $\|D(D^{-1}\xi)\| = \|(1-\Pi_0)\xi\| \leq \|\xi\|$. Thus $\|D^{-1}\xi\|_1 = \sqrt{\|D^{-1}\xi\|^2 + \|D(D^{-1}\xi)\|^2} \leq \sqrt{\|D^{-1}\|^2+1}\,\|\xi\|$.

This proves that D^{-1} gives rise to a cont. op. from $\mathcal{H}$ to $\mathcal{H}_1$. The proof is complete. $\square$

Lemma 11.2: Let $a \in A$. Then

(i) The action of a on $\mathcal{H}$ induces a continuous endomorphism of $\mathcal{H}_1$.

(ii) $[D,a]|_{\mathcal{H}_1}$ is a compact operator from $\mathcal{H}_1$ to $\mathcal{H}$.

By assumption $a(\text{dom} D) \subset \text{dom} D$, so a acts on $\mathcal{H}_1$. Let $\xi \in \mathcal{H}_1$. Then

$$\|Da\xi\| = \|aD\xi - [D,a]\xi\| \leq \|a\| \, \|D\xi\| + \|[D,a]\| \, \|\xi\|.$$

Thus,

$$\|a\xi\|_1 = \sqrt{\|a\xi\|^2 + \|Da\xi\|^2} \leq \sqrt{\|a\|^2\|\xi\|^2 + (\|a\| + \|[D,a]\|)^2 \|\xi\|^2} \leq \sqrt{\|a\|^2 + (\|a\| + \|[D,a]\|)^2} \cdot \|\xi\|$$

This shows that a acts continuously on $\mathcal{H}_1$.

In addition, as $[D,a]|_{\mathcal{H}_1}$ agrees with the composition,

$$\mathcal{H}_1 \underset{\text{compact}}{\overset{\iota}{\hookrightarrow}} \mathcal{H} \underset{\text{bounded}}{\xrightarrow{[D,a]}} \mathcal{H},$$

we see that $[D,a]|_{\mathcal{H}_1}$ is a compact operator from $\mathcal{H}_1$ to $\mathcal{H}$. The proof is complete.

Let $q \in \mathbb{N}$. The Hilbert space $\mathcal{H}^q = \mathcal{H} \otimes \mathbb{C}^q$ is $\mathbb{Z}_2$-graded,

$$\mathcal{H}^q = (\mathcal{H}^+)^q \oplus (\mathcal{H}^-)^q.$$

As the action of A on $\mathcal{H}$ preserves the $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, the above $\mathbb{Z}_2$-grading of $\mathcal{H}^q$ is preserved by the action of $M_q(A)$ given by

$$(a_{ij}) \cdot (\xi_j) := \Big(\sum_j a_{ij}\xi_j\Big) \qquad \forall a = (a_{ij}) \in M_q(A),$$
$$\forall \xi = (\xi_j) \in \mathcal{H}^q.$$

Let $e \in M_q(A)$ be a projection, $e^2 = e^* = e$, and define

$$e\mathcal{H}^q = \{e\xi;\ \xi \in \mathcal{H}^q\} = \{\xi \in \mathcal{H}^q;\ e\xi = \xi\}.$$

This is a closed subspace of $\mathcal{H}^q$, and hence it is a Hilbert space with respect to the induced inner product.

Furthermore, as the action of e preserves both $(\mathcal{H}^+)^q$ and $(\mathcal{H}^-)^q$ we see that

$$e(\mathcal{H}^\pm)^q = \{e\xi;\ \xi \in (\mathcal{H}^\pm)^q\} = \{\xi \in (\mathcal{H}^\pm)^q;\ e\xi = \xi\}.$$

Thus $e(\mathcal{H}^+)^q$ and $e(\mathcal{H}^-)^q$ both are closed subspaces of $e\mathcal{H}^q$ and we have the $\mathbb{Z}_2$-grading,

$$(1) \qquad e\mathcal{H}^q = e(\mathcal{H}^+)^q \oplus e(\mathcal{H}^-)^q.$$

In addition, as by Lemma 11.2 A acts continuously on $\mathcal{H}_1$, we can similarly see that $e\mathcal{H}_1^q$ and $e(\mathcal{H}_1^\pm)^q$ are closed subspaces of $\mathcal{H}_1^q$, and hence are Hilbert spaces with respect to the respective induced inner products.

<u>Def.</u>: D_e is the unbounded operator of $e\mathcal{H}^q$ w/ domain $e\mathcal{H}_1^q$ defined by

$$D_e\xi := e(D \otimes I_q)\xi = e(D\xi_j) \qquad \forall \xi = (\xi_j) \in e\mathcal{H}_1^q$$

Notice that with respect to the $\mathbb{Z}_2$-grading (1) D_e takes the form,

$$D_e = \begin{pmatrix} 0 & D_e^- \\ D_e^+ & 0 \end{pmatrix}, \quad D_e^\pm := e(D^\pm \otimes I_q) : e(\mathcal{H}_1^\pm)^q \longrightarrow e(\mathcal{H}_1^\mp)^q.$$

We shall regard $D_e^\pm$ as an unbounded operator from $e(\mathcal{H}^\pm)^q$ to $e(\mathcal{H}^\mp)^q$ w/ domain $e(\mathcal{H}_1^\pm)^q$.

Lemma II.3: (i) D_e is selfadjoint.

(ii) $(D_e^+)^* = D_e^-$.

Proof: As $D\otimes I_q$ w/ domain $\mathcal{H}_1^q$ is selfadjoint, upon replacing D by $D\otimes I_q$, $\mathcal{H}$ by $\mathcal{H}^q$ and A by $M_q(A)$ we may assume that $q=1$.

Let $\xi \in e\mathcal{H}_1$. Then, for all $\eta \in e\mathcal{H}_1$,

$$\langle D_e\xi, \eta\rangle = \langle eD\xi, \eta\rangle = \langle D\xi, e\eta\rangle = \langle D\xi, \eta\rangle = \langle \xi, D\eta\rangle$$
$$= \langle e\xi, D\eta\rangle = \langle \xi, eD\eta\rangle = \langle \xi, D_e\eta\rangle.$$

Thus D_e is symmetric, i.e., we have the inclusion of graphs,

$$G(D_e) \subset G(D_e^*), \text{ and hence } G(D_e) = G(D_e^*) \cap (\operatorname{dom} D_e \times e\mathcal{H})$$

By definition,

$$G(D_e^*) = \{(\xi, \eta) \in e\mathcal{H} \times e\mathcal{H};\ \langle \eta, \zeta\rangle = \langle \xi, D\zeta\rangle \quad \forall \zeta \in e\mathcal{H}_1\}.$$

Let $(\xi,\eta) \in G(D_e^*)$. Recall that $[D,e]$ is bounded and hence has domain $\mathcal{H}$. Let $\zeta \in \mathcal{H}_1$. Then

$$\langle \eta, \zeta\rangle = \langle e\eta, \zeta\rangle = \langle \eta, e\zeta\rangle.$$

As $e\zeta \in e\mathcal{H}_1$, we see that

$$\langle \eta, \zeta\rangle = \langle \xi, De\zeta\rangle = \langle \xi, (eD + [D,e])\zeta\rangle$$
$$= \langle \xi, eD\zeta\rangle + \langle \xi, [D,e]\zeta\rangle$$
$$= \langle e\xi, D\zeta\rangle + \langle [D,e]^*\xi, \zeta\rangle$$
$$= \langle \xi, D\zeta\rangle + \langle [D,e]^*\xi, \zeta\rangle$$

Thus,

$$\langle \eta - [D,e]^*\xi, \zeta\rangle = \langle \xi, D\zeta\rangle \quad \forall \zeta \in \mathcal{H}_1,$$

that is,

$$(\xi, \eta - [D,e]^*\xi) \in G(D^*) = G(D).$$

In particular, we see that $\xi \in \operatorname{dom} D_e$, and hence

$$(\xi,\eta) \in G(D_e^*) \cap (\operatorname{dom} D_e \times e\mathcal{H}) = G(D_e).$$

This proves that $G(D_e^*) = G(D_e)$, that is, D_e is selfadjoint.

Next, we have

$$G((D_e^+)^*) = \{(\xi,\eta) \in (e\mathcal{H}^-)\times(e\mathcal{H}^+);\ \langle \eta, \zeta\rangle = \langle \xi, D_e^+\zeta\rangle \quad \forall \zeta \in e\mathcal{H}_1^+\}$$
$$= \{(\xi,\eta) \in (e\mathcal{H}^-)\times(e\mathcal{H}^+);\ \langle \eta, \zeta\rangle = \langle \xi, D_e\zeta\rangle \quad \forall \zeta \in e\mathcal{H}_1\}.$$

Thus,

$$G((D_e^+)^*) = G(D_e^*) \cap (\mathcal{H}_1^- \times \mathcal{H}_1^+) = G(D_e) \cap (\mathcal{H}_1^- \times \mathcal{H}_1^+) = G(D_e^-).$$

This shows that $(D_e^+)^* = D_e^-$. The proof is complete. □

Notice that if we regard D_e as an operator from $e\mathcal{H}_1^q$ to $e\mathcal{H}^q$, then D_e is continuous, since $D \otimes I_q$ is continuous from $\mathcal{H}_1^q$ to $\mathcal{H}^q$ and it follows from Lemma 11.2 that e acts continuously on $\mathcal{H}_1^q$.

<u>Lemma 11.4</u>: (i) Seen as an operator from $e\mathcal{H}_1^q$ to $e\mathcal{H}^q$, the operator D_e is Fredholm.
(ii) Seen as an operator from $e(\mathcal{H}_1^\pm)^q$ to $e(\mathcal{H}^\mp)^q$ the operator $D_e^\pm$ is Fredholm and

$$\operatorname{ind} D_e^+ = \dim\ker D_e^+ - \dim\ker D_e^- .$$

<u>Proof.</u> As in the proof of Lemma 11.3 we may assume that $q=1$.
The operator $eD^{-1}: e\mathcal{H} \to e\mathcal{H}_1$ is continuous and, on $e\mathcal{H}_1$, we have

$$\begin{aligned} D_e . eD^{-1} &= e(De)D^{-1} = e(eD + [D,e])D^{-1} \\ &= e^2 \overbrace{DD^{-1}}^{=1-\Pi_0} + e[D,e]D^{-1} \\ &= \underbrace{e}_{\substack{\| \\ \mathrm{id\ on\ } e\mathcal{H}^q}} - \overset{\text{finite rank}}{e\Pi_0} + \underbrace{e[D,e]}_{\text{bounded}} . \underset{\text{compact}}{D^{-1}} \\ &= 1 \mod \mathcal{K}(e\mathcal{H}). \end{aligned}$$

In addition, on $e\mathcal{H}$, we have

$$eD^{-1}.D_e = eD^{-1}(De - [D,e]) = \underbrace{e}_{\substack{\| \\ \mathrm{id\ on\ } e\mathcal{H}_1}} - \underset{\text{finite rank}}{e\Pi_0} - e\overset{\mathcal{L}(\mathcal{H},\mathcal{H}_1)}{D^{-1}} . \overbrace{[D,e]|_{\mathcal{H}_1}}^{\mathcal{K}(\mathcal{H}_1,\mathcal{H}) \text{ by Lemma 11.2}} = \mathrm{id}_{e\mathcal{H}_1} \mod \mathcal{K}(e\mathcal{H}_1).$$

This shows that D_e is invertible modulo compact operators, and hence is Fredholm.
Observe that with respect to the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$

$$D^{-1} = \begin{pmatrix} 0 & (D^+)^{-1} \\ (D^-)^{-1} & 0 \end{pmatrix}.$$

Therefore, with respect to the splittings $e\mathcal{H} = e\mathcal{H}^+ \oplus e\mathcal{H}^-$ and $e\mathcal{H}_1 = e\mathcal{H}_1^+ \oplus e\mathcal{H}_1^-$,

$$eD^{-1} = \begin{pmatrix} 0 & e(D^+)^{-} \\ e(D^-)^{-1} & 0 \end{pmatrix}.$$

Therefore, the fact that eD^{-1} is an inverse of D_e modulo compact operators implies that $e(D^\mp)^{-1}$ is an inverse of $D_e^\pm$ modulo compact operators. Therefore, $D_e^\pm$ is Fredholm.

If we regard D_e^+ as a continuous operator from $e\mathcal{H}_1^+$ from $e\mathcal{H}^-$, then D_e^+ has adjoint $(D_e^+)^{(*)}$ defined as the bounded operator from $e\mathcal{H}^-$ to $e\mathcal{H}_1^+$ s.t.

$$\langle (D_e^+)^{(*)}\xi, \eta\rangle = \langle \xi, D_e^+\eta\rangle \quad \forall \xi\in e\mathcal{H}^- \ \forall \eta \in e\mathcal{H}_1^+.$$

Thus, if $\xi \in e\mathcal{H}^-$, then

$$\begin{aligned}\xi \in \ker (D_e^+)^{(*)} &\iff \langle \xi, D_e^+\eta\rangle = 0 \quad \forall \eta \in e\mathcal{H}_1^+\\ &\iff (\xi, 0) \in G((D_e^+)^*)\\ &\iff \xi \in \ker (D_e^+)^*.\end{aligned}$$

As $(D_e^+)^* = D_e^-$, we see that $\ker (D_e^+)^{(*)} = \ker D_e^-$, and hence

$$\begin{aligned}\operatorname{ind} D_e^+ &= \dim \ker D_e^+ - \dim \ker (D_e^+)^{(*)}\\ &= \dim \ker D_e^+ - \dim \ker D_e^-.\end{aligned}$$

The lemma is proved. $\square$

Def.: The index of D_e is

$$\begin{aligned}\operatorname{ind} D_e &= \operatorname{ind} D_e^+\\ &= \dim \ker D_e^+ - \dim \ker D_e^-.\end{aligned}$$

Lemma 11.4: Let $u \in M_q(A)$ be unitary. Then

$$\operatorname{ind} D_{u^*eu} = \operatorname{ind} D_e.$$

Proof. As in the proofs of Lemma 11.3 and Lemma 11.4 we may assume that $q=1$.

Set $\tilde{e} = u^*eu$. Then the actions of u and u^* give rise to graded bounded operators

$$u : \tilde{e}\mathcal{H} \to e\mathcal{H} \quad \text{and} \quad u^* : e\mathcal{H} \to \tilde{e}\mathcal{H},$$

which are inverses of each other. (Here the fact that u is graded means that $u(\tilde{e}\mathcal{H}^{\pm}) \subset e\mathcal{H}^{\pm}$.)

Likewise, by using Lemma 11.2, we see that the actions of u and u^* also give rise to bounded graded operators

$$u : \tilde{e}\mathcal{H}_1 \to e\mathcal{H}_1 \quad \text{and} \quad u^* : e\mathcal{H}_1 \to \tilde{e}\mathcal{H}_1,$$

which are inverses of each other. Then u^*D_eu is a continuous operator from $\tilde{e}\mathcal{H}_1$ to $\tilde{e}\mathcal{H}$.

On $\tilde{e}\mathcal{H}_1$ we have

$$\begin{aligned}u^*D_eu &= u^*eDu = u^*e(uD + [D,u])\\ &= \underbrace{u^*eu}_{=\tilde{e}}D + u^*e\underbrace{[D,u]|_{\mathcal{H}_1}}_{\text{Compact from } \mathcal{H}_1 \text{ to } \mathcal{H} \text{ by Lemma 11.2}}\\ &= D_{\tilde{e}} \ \cancel{\text{mod}}\ \mathcal{K}(\tilde{e}\mathcal{H}_1, \tilde{e}\mathcal{H})\end{aligned}$$

Observe also that, with respect to the splittings, $\tilde{e}\mathcal{H}_1 = \tilde{e}\mathcal{H}_1^+ \oplus \tilde{e}\mathcal{H}_1^-$ and $\tilde{e}\mathcal{H} = \tilde{e}\mathcal{H}^+ \oplus \tilde{e}\mathcal{H}^-$,

$$u^* D_e u = \begin{pmatrix} 0 & u^* D_e^- u \\ u^* D_e^+ u & 0 \end{pmatrix}.$$

Therefore, we see that

$$u^* D_e^+ u = D_{\tilde{e}}^+ \mod \mathcal{K}(\tilde{e}\mathcal{H}_1^+, \tilde{e}\mathcal{H}^-).$$

Thus,

$$\operatorname{ind} D_{\tilde{e}}^+ = \operatorname{ind}(u^* D_e^+ u) = \operatorname{ind} D_e^+,$$

and hence $\operatorname{ind} D_{\tilde{e}}$ and $\operatorname{ind} D_e$ agree. The proof is complete. $\square$.

The above lemma shows that $\operatorname{ind} D_e$ depends only on the unitary equivalence class of e. Furthermore, if $e_1 \in M_{q_1}(A)$ and $e_2 \in M_{q_2}(A)$ are projections, then it is not hard to see that

$$\begin{aligned} \operatorname{ind} D_{e_1 \oplus e_2} = \operatorname{ind} D_{e_1 \oplus e_2}^+ = \operatorname{ind}(D_{e_1}^+ \oplus D_{e_2}^+) &= \operatorname{ind} D_{e_1}^+ + \operatorname{ind} D_{e_2}^+ \\ &= \operatorname{ind} D_{e_1} + \operatorname{ind} D_{e_2}. \end{aligned}$$

Therefore, we obtain

<u>Prop</u>. There exists a unique additive map,

$$\operatorname{ind}_D : K_0(A) \longrightarrow \mathbb{Z},$$

such that, for all projections $e \in M_q(A)$,

$$\operatorname{ind}_D [e] = \operatorname{ind} D_e.$$

7. The Connes-Chern Character (Connes): In the sequel:

$(A,\mathcal{H},D)$ = spectral triple, $\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-$, $\gamma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Assumption: $(A,\mathcal{H},D)$ is p-summable, i.e., $\mu_n(D^{-1})=O(n^{-1/p})$

Consequence: $|D|^{-1}\in\mathcal{L}^q\ \forall q>p$

In addition, we set $F=D|D|^{-1}$. Notice that $F^2=1-\Pi_0$, Π_0 = orthogonal projection onto $\ker D$ = finite rank projection.

Lemma: $[F,a]\in\mathcal{L}^q\ \forall a\in A\ \forall q>p$.

Consequence: If $m+1>p$, then $[F,a^0]\ldots[F,a^m]\in\mathcal{L}^1\ \forall a^j\in A$.

Set $n_0=\left[\frac{p+1}{2}\right]$.

Def: For $n\geq n_0$ we denote by τ_{2n} the cochain in $C^{2n}(A)$ defined by

$$\tau_{2n}(a^0,\ldots,a^{2n}):=\frac{1}{2}\cdot\frac{n!}{(2n)!}\operatorname{Tr}\left\{\gamma F[F,a^0]\ldots[F,a^{2n}]\right\}\quad\forall a^j\in A.$$

Rmk: If $n\geq n_0+1$, then

$$\tau_{2n}(a^0,\ldots,a^{2n})=\frac{n!}{(2n)!}\operatorname{Tr}\left\{\gamma a^0[F,a^1]\ldots[F,a^{2n}]\right\}\quad\forall a^j\in A.$$

Lemma τ_{2n} is a normalized cyclic cocycle.

We can identify τ_{2n} with the even cocycle $(0,\ldots,\underset{(2n+1)\text{th slot}}{\tau_{2n}},0,\ldots)\in C^{ev}(A)$.

For $n\geq n_0$, define

$$\nu_{2n+1}(a^0,\ldots,a^{2n+1}):=\frac{(n+1)!}{(2n+2)!}\operatorname{Tr}\left\{\gamma a^0F[F,a^1]\ldots[F,a^{2n+1}]\right\}\quad\forall a^j\in A.$$

Lemma. $b\nu_{2n+1}=-\tau_{2n+2}$ and $B\nu_{2n+1}=\tau_{2n}$.

Therefore, upon identifying ν_{2n+1} with the odd cochain $(0,\ldots,0,\nu_{2n+1},0,\ldots)\in C^{odd}(A)$, we have

$$\tau_{2n}-\tau_{2n+1}=B\nu_{2n+1}+b\nu_{2n+1}.$$

$\Rightarrow$ the class of τ_{2n} in $HC^{ev}(A)$ does not depend on n, $n\geq n_0$.

Def. The class of τ_{2n} in $HC^{ev}(A)$ is called the Connes-Chern character of $(A,\mathcal{H},D)$ and is denoted $Ch_*(D)$.

Thm. (Connes) (1) For all $e\in M_q(A)$, $e^*=e^2=e$, we have

$$\operatorname{ind}D_e=(-1)^n\operatorname{Tr}\left\{\gamma e[F,e]^{2n}\right\}=\langle\tau_{2n},e\rangle.$$

(2) The Connes-Chern character computes the index map of D, namely,

$$\operatorname{ind}_D[e]=\langle Ch_*D,e\rangle\quad\forall e\in K_0(A)$$

8. Hochschild Character Theorem (Connes):

The cyclic cocycle τ_{2m} is hard to compute in practice. ~~We n~~

$\Rightarrow$ We would like to find a representative given by local formulas, i.e., formulas not involving the operator trace Tr.

Assume $p = 2m$, $m \in \mathbb{N}_0 \Rightarrow n_0 = m$ and $\mu_n(|D|^{-2m}) = \mu_n(|D|^{-1})^{2m} = O(n^{-1})$.

Define

$$\varphi_{2m}(a^0, \dots, a^{2m}) := \frac{1}{(2m)} \frac{m!}{(2m)!} \mathrm{Tr}_\omega\left\{ \gamma a^0 [D, a^1] \dots [D, a^{2m}] |D|^{-2m} \right\} \quad \forall a^j \in A.$$

Thm. (Connes) φ_{2m} is a Hochschild cocycle which differs from τ_{2m} by a Hochschild coboundary.

Thus φ_{2m} can be thought as a representative of $Ch_* D$ in Hochschild cohomology.

9. The Local Index Formula in NCG (Connes-Moscovici, '95):

- We would like to get a local representative of $Ch_* D$ in $HC^\infty(A)$, not just in Hochschild cohomology.
- Need to extend the Dixmier trace $\Rightarrow$ need a NC residue trace for $(A, \mathcal{H}, D)$.
- Regularity:

$\delta(T) := [|D|, T]$ $\quad \delta$ = unbounded derivation on $\mathcal{L}(\mathcal{H})$

$\qquad \mathrm{dom}\,\delta := \{ T \in \mathcal{L}(\mathcal{H});\ [|D|, T] \text{ bounded} \}$.

Def.: $(A, \mathcal{H}, D)$ is said to be regular when, for all $a \in A$,

$\qquad a$ and $[D, a]$ are in $\bigcap_{n \geq 1} \mathrm{dom}\,\delta^n$.

- Dimension Spectrum:

$\mathcal{B}$ = algebra generated by γ, $\delta^n(T)$, $n \geq 0$, $T = a$ or $[D, A]$, $a \in A$

Let $P \in \mathcal{B}$. For $\mathrm{Re}\, z >> 1$ $\quad P|D|^{-z} \in \mathcal{L}^1 \Rightarrow \zeta_P(z) := \mathrm{Tr}[P|D|^{-z}]$ well-defined.

Ω_P = maximal open set to which $\zeta_P(z)$ analytically extends.

Def. The dimension spectrum of $(A, \mathcal{H}, D)$ is

$$\Sigma := \bigcup_{P \in \mathcal{B}} (\mathbb{C} \setminus \Omega_P).$$

Rk. p-summability $\Rightarrow$ $\Sigma \subset \{ \mathrm{Re}\, z \leq p \}$.

Assumptions:

1. $(A, \mathcal{H}, D)$ is regular 2. Σ is discrete and simple, i.e., Σ is a discrete subset of $\mathbb{C}$ and the singularities of the $\zeta_P(z)$ on Σ are at most simple poles.

- **ΨDOs and NC Residue Trace:**

For $s \geq 0$ let $\mathcal{H}^s = \mathrm{dom}(|D|)^s$ with $\|\xi\|_s = \||D|^s\xi\|$.
For $s<0$: $\mathcal{H}^s$ = dual of $\mathcal{H}^{-s}$.
In addition set $\mathcal{H}^\infty := \bigcap_{s\geq 0} \mathcal{H}^s$ w/ topology defined by the semi-norms $\|.\|_s$, $s \geq 0$.

Lemma: If $T \in \bigcap_{\mu>0} \mathrm{dom}\, \delta^\mu$, then $T \in \mathcal{L}(\mathcal{H}^s, \mathcal{H}^s)$ $\forall s \in \mathbb{R}$.

For $m \in \mathbb{R}$ set $OP^m := \{T \in \mathcal{L}(\mathcal{H}^\infty);\ |D|^{-m} T \in \bigcap_{\mu\geq 0} \mathrm{dom}\, \delta^\mu\}$

Lemma: $|D|^z \in OP^{\mathrm{Re}(z)}$ $\forall z \in \mathbb{C}$.

Def: $\Psi_D^m(A)$, $m \in \mathbb{C}$, consists of op. $P \in \mathcal{L}(\mathcal{H}^\infty)$ such that
$$P \sim \sum_{j\geq 0} b_{m-j} |D|^{m-j}, \quad b_{m-j} \in \mathcal{B}$$
where $\sim$ means that
$$\forall N \in \mathbb{N}\ \exists J \in \mathbb{N} \text{ s.t. } P - \sum_{j\leq J} b_{m-j} |D|^{m-j} \in OP^{-N}$$

Lemma. If $P_j \in \Psi_D^{m_j}(A)$, $j=1,2$, then $P_1 P_2 \in \Psi_D^{m_1+m_2}(A)$.

Lemma: (1) The formula
$$⨍ P := \mathrm{Res}_{z=0} \mathrm{Tr}\, P |D|^{-z}$$
defines a linear functional on each space $\Psi_D^m(A)$.
(2) ⨍ is a trace, in the sense that
$$⨍ P_1 P_2 = ⨍ P_2 P_1 \quad \forall P_j \in \Psi_D^{m_j}(A)$$
(3) ⨍ is local: it vanishes on $\Psi_D^m(A)$ w/ $\mathrm{Re}\, m < -p$.

- **The CM cocycle:**

For $T \in \mathcal{L}(\mathcal{H}^\infty)$ and $l \in \mathbb{N}$ we set
$$T^{[l]} = \underbrace{[D^2, [D^2, \ldots, [D^2}_{l\text{-times}}, T] \ldots]]$$

Thm. (Connes–Moscovici) (1) The following formula defines a normalized even cocycle $\varphi_{CM} = (\varphi_{2n})$ in the (b, B)-complex of A:

- For $n=0$:

$$\varphi_0(a^0) = \frac{1}{2}\operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[a^0\left(|D|^{-2z} + \Pi_0\right)\right]\right\}, \quad \Pi_0 = \text{orth. proj. onto } \ker D.$$

- For $n \geq 1$:

$$\varphi_{2n}(a^0, \dots, a^{2n}) = \sum_{\alpha \in \mathbb{N}_0^{2n}} c_{n,\alpha} ⨍ a^0 [D,a^1]^{[\alpha_1]} \cdots [D,a^{2n}]^{[\alpha_{2n}]} |D|^{-2(n+|\alpha|)},$$

where the sum is actually finite and

$$c_{n,\alpha} = \frac{(-1)^{|\alpha|}}{2} \cdot \frac{\Gamma(|\alpha|+n)}{\alpha!\,(\alpha_1+1)(\alpha_1+\alpha_2+2)\cdots(\alpha_1+\cdots+\alpha_{2n}+2n)}.$$

(2) The class of φ_{CM} in $HC^\infty(A)$ is equal to the Connes-Chern character $Ch_* D$.

(3) For all $e \in M_q(A)$, $e^2 = e^* = e$,

$$\operatorname{ind} D_e = \langle \varphi_{CM}, e \rangle.$$

Rk: (1) $⨍$ local $\Rightarrow$ formula for φ_{2n}, $n \geq 1$, is local

$\Rightarrow$ (3) is a local index formula.

(2) The cocycle φ_{CM} is called the CM cocycle of $(A, \mathcal{H}, D)$.

11.7. The CM Cocycle of a Dirac Spectral Triple

In this section, we explain how to compute. computation of the CM cocycle of a Dirac spectral triple given in [**Po**].

Let (M^n, g) be an even dimensional compact oriented Riemannian spin manifold with spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$ and Dirac operator $\not{D}_M : C^\infty(M, \not{S}) \rightarrow C^\infty(M, \not{S})$. We shall use the same notation as in Chapter 8 without any further notice.

Let us explain why the Dirac spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ satisfies the assumptions of the local index formula in noncommutative geometry.

As $\not{D}_M$ is a ΨDO of order -1, the results of Chapter 7 show that the characteristic values of $\not{D}_M^{-1}$ satisfy

$$\mu_k(\not{D}_M^{-1}) = \mathrm{O}(k^{-\frac{1}{n}}),$$

that is, the spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ is n-summable.

Consider the derivation $\delta(T) := [|\not{D}_M|, T]$ with domain

$$\operatorname{dom}\delta := \left\{ T \in \mathcal{L}(L^2(M, \not{S});\ [|\not{D}_M|, T] \in \mathcal{L}(L^2(M, \not{S}) \right\}.$$

Lemma 11.7.1. *$\Psi^0(M, \not{S})$ is contained in* $\operatorname{dom}\delta$ *and*

$$\delta\left(\Psi^0(M, \not{S})\right) \subset \Psi^0(M, \not{S}).$$

Proof. By the results of Chapter 6 on the complex powers of an elliptic ΨDO,

$$\sigma_1(|\not{D}_M|) = \sigma_1\left(\sqrt{\not{D}_M^2}\right) = \sqrt{\sigma_2(\not{D}_M^2)}.$$

Recall that $\sigma_1(\not{D}_M)(x,\xi) = ic(\xi)$ (cf. Chapter 8), so we have

$$\sigma_2(\not{D}_M^2)(x,\xi) = (\sigma_1(\not{D}_M)(x,\xi))^2 = (ic(\xi))^2 = \langle \xi, \xi \rangle_g = |\xi|_g^2,$$

where $\langle \cdot, \cdot \rangle_g$ and $|\cdot|_g$ are the inner product and norm on the fibers of T^*M defined by the Riemmanian metric g. Thus,

$$\sigma_1(|\not{D}_M|)(x,\xi) = \sqrt{\sigma_2(\not{D}_M^2)(x,\xi)} = |\xi|_g.$$

The above formula shows that $\sigma_1(|\not{D}_M|)(x,\xi)$ is scalar, and hence commutes with all principal symbols. Therefore, if $P \in \Psi^0(M, \not{S})$, then the commutator $[\not{D}_M, P] = |\not{D}_M|P - P|\not{D}_M|$ is a ΨDO of order ≤ 1 whose symbol of order 1 is equal to

$$\sigma_1(|\not{D}_M|)\sigma_0(P) - \sigma_0(P)\sigma_1(|\not{D}_M|) = 0.$$

In other words, $[\not{D}_M, P]$ is a zeroth order ΨDO. As the zeroth order ΨDOs are bounded, this implies that $\Psi^0(M, \not{S})$ is contained in $\operatorname{dom}\delta$ and $\delta\left(\Psi^0(M, \not{S})\right)$ is contained in $\Psi^0(M, \not{S})$. The lemma is proved. □

It follows from Lemma 11.7.1 that $\Psi^0(M, \not{S})$ is contained in $\cap_{j\geq 1}\operatorname{dom}\delta^j$ and

$$\delta^j\left(\Psi^0(M, \not{S})\right) \subset \Psi^0(M, \not{S}) \qquad \forall j \in \mathbb{N}. \tag{11.1}$$

Observe that if $f \in C^\infty(M)$, then both f and $[D, f] = c(df)$ are zeroth order ΨDOs, and hence belong to $\cap \operatorname{dom}\delta^j$. This shows that the spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ is regular.

Define

$$\Sigma := \{k \in \mathbb{Z}; k \leq n\}.$$

It follows from (11.1) that the algebra $\mathcal{B}$ constructed in the previous section is contained in $\Psi^0(M, \not{S})$. If $P \in \Psi^0(M, \not{S})$, then by the results of Chapter 6 the function $z \to \operatorname{Tr}[P|\not{D}_M|^{-z}]$ has a meromorphic continuation to $\mathbb{C}$ which is holomorphic on $\mathbb{C}\backslash\Sigma$ and on Σ has at worst simple pole singularities. We then deduce that the spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ has a simple and discrete dimension spectrum contained in Σ.

In addition, it can be shown that, for all $m \in \mathbb{C}$, the space $\Psi^m_{\not{D}_M}(C^\infty(M))$, introduced in the previous section, is contained in $\Psi^m(M, \not{S})$ and, with the notation of the previous section, for all $P \in \Psi^m_{\not{D}_M}(C^\infty(M))$,

$$\int\!\!\!\!\!\!- P = \operatorname{Res}_{z=0} \operatorname{Tr}\left[P|\not{D}_M|^{-z}\right] = \operatorname{Res} P,$$

where $\operatorname{Res} P$ is the noncommutative residue of P (cf. Chapter 6).

As the Dirac spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ satisfies the assumptions of the local index formula in noncommutative geometry, its CM cocycle $\varphi_{\mathrm{CM}} = (\varphi_{2k})$ makes sense and is given by the following formulas:

- If $k = 0$, then, for all $f^0 \in C^\infty(M)$,

$$\varphi_0(f^0) = \operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0(|\not{D}_M|^{-z} + \Pi_0)\right]\right\},$$

  where Π_0 is the orthogonal projection onto $\ker \not{D}_M$.
- If $k \geq 1$, then, for all $f^j \in C^\infty(M)$,

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \sum_\alpha c_{k,\alpha} \int\!\!\!\!\!\!- \gamma f^0[D, f^1]^{[\alpha_1]} \ldots [D, f^{2k}]^{[\alpha_{2k}]}|\not{D}_M|^{-2(|\alpha|+k)}, \tag{11.2}$$

  where $\Gamma(|\alpha|+k)c^{-1}_{k,\alpha} = 2(-1)^{|\alpha|}\alpha!(\alpha_1+1)\cdots(\alpha_1+\cdots+\alpha_{2k}+2k)$ and the symbol $T^{[j]}$ denotes the j-th iterated commutator with $\not{D}^2_M$.

In order to compute the cochains φ_{2k} we will need a differentiable version of the local index theorem as follows.

Definition 11.7.2. *We say that an operator $Q \in \Psi^*_{\mathrm{v}}(M \times \mathbb{R}, \not{S})$ has Getzler order m when, for all $x_0 \in M$, in any normal coordinates centered at x_0 the operator Q has Getzler order m in the sense of Definition 8.12.13.*

Proposition 11.7.3. *Let $\not{P} : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ be a differential operator of Getzler order m and, for $t > 0$, let us denote by $h_t(x, y)$ the kernel of $\not{P}e^{-t\not{D}^2_M}$. Then, as $t \to 0^+$, there is an asymptotics in $C^\infty(M, |\Lambda|(M))$ of the form*

$$\operatorname{Str}_{\not{S}_x} h_t(x,x) = \begin{cases} \mathrm{O}(t^{\frac{-m+1}{2}}) & \text{if } m \text{ is odd,} \\ t^{\frac{-m}{2}} B_0(\not{D}^2_M, \not{P})(x) + \mathrm{O}(t^{\frac{-m}{2}+1}) & \text{if } m \text{ is even,} \end{cases}$$

where, in normal coordinates centered at x_0,

$$B_0(\not{D}^2_M, \not{P})(x_0) = (-2i)^{\frac{n}{2}}\left[\not{P}_{(m)}G_R(0,1)\right]^{(n)},$$

where $\not{P}_{(m)}$ is the model operator of $\not{P}$ and $G_R(x,t)$ is the fundamental solution of $H_R + \partial_t$ given by Lemma 8.12.21.

Proof. As in the proof of Proposition 8.12.12,

$$h_t(x,y) = K_{\not{P}(\not{D}^2_M+\partial_t)^{-1}}(x,y,t).$$

By Lemma 8.12.18 and Lemma 8.12.19, in normal coordinates, $P(D_W^2 + \partial_t)^{-1}$ has Getzler order $m-2$ and its model operator is

$$Q_{(m-2)} = P_{(m)}(H_R + F^W + \partial_t)^{-1}.$$

Thus,

$$K_{Q_{(m-2)}}(x,0,t) = P_{(m)x} K_{(H_R+\partial_t)^{-1}}(x,0,t) = (P_{(m)} G_R)(x,t).$$

The proposition then follows from Proposition 8.12.12 and Lemma 8.12.17. $\square$

THEOREM 11.7.4. *The CM cocycle $\varphi_{\mathrm{CM}} = (\varphi_{2k})$ is given by*

(11.3)

$$\varphi_{2k}(f^0,\ldots,f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!}\int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M)^{(n-2k)}, \quad f^j \in C^\infty(M),$$

where $\hat{A}(R^M)$ is the $\hat{A}$-form of the Riemann curvature R^M.

PROOF. Let $k \in \mathbb{N}$ and $f^0, \cdots, f^{2k}$ in $C^\infty(M)$. For all $\alpha \in \mathbb{N}_0^{2k}$ we set

$$P_\alpha = f^0 [D_M, f^1]^{[\alpha_1]} \cdots [D_M, f^{2k}]^{[\alpha_{2k}]} = f^0 c(df^1)^{[\alpha_1]} \cdots c(df^{2k})^{[\alpha_{2k}]}.$$

Then

$$\varphi_{2k}(f^0,\ldots,f^{2k}) = \sum c_{k,\alpha} ⨍ \gamma P_\alpha |D_M|^{-2(|\alpha|+k)}. \tag{11.4}$$

Observe also that

$$\begin{aligned} ⨍ \gamma P_\alpha |D_M|^{-2(|\alpha|+k)} &= 2\,\mathrm{Res}_{z=0}\,\mathrm{Tr}\left[\gamma P_\alpha |D_M|^{-2(|\alpha|+k)} . |D_M|^{-2z}\right] \\ &= 2\,\mathrm{Res}_{z=0}\,\mathrm{Str}\left[P_\alpha |D_M|^{-2(|\alpha|+k+z)}\right]. \end{aligned} \tag{11.5}$$

CLAIM. For $t > 0$ let $h_{\alpha,t}(x,y)$ be the kernel of $P_\alpha e^{-tD_M^2}$. Then, as $t \to 0^+$, there is an asymptotics in $C^\infty(M, |\Lambda|(M))$ of the form

(11.6)

$$\mathrm{Str}_{\$}\, h_{\alpha,t}(x,x) = \begin{cases} \mathrm{O}(t^{-(k+|\alpha|)+1}) & \text{if } \alpha \neq 0, \\ \frac{t^{-k}}{(2i\pi)^{\frac{n}{2}}} \int_M f^0 df^1 \wedge \ldots \wedge df^{2k} \wedge \hat{A}(R^M)^{(n-2k)} + \mathrm{O}(t^{-k+1}) & \text{if } \alpha = 0. \end{cases}$$

PROOF OF THE CLAIM. In synchronous normal coordinates $c(df^j)$ and D^2 have respective Getzler orders 1 and 2 and respective model operators $df^j(0)$ and $H_R = -\sum(\partial_i - R_{ij}^M(0)x^j)^2$. Therefore, by Lemma 8.12.18 the operator P_α has Getzler order $\leq 2(k+|\alpha|)$ and

$$P_\alpha = c[f^0(0)df^1(0)^{[\alpha_1]} \wedge \cdots \wedge df^{2k}(0)^{[\alpha_{2k}]}] + \mathrm{O}_G(2(k+|\alpha|)-1),$$

where $T^{[j]}$ is the j-th iterated commutator of T with H_R.

Notice that $[H_R, df^j(0)] = 0$, so if $\alpha \neq 0$ then $P_\alpha Q$ has Getzler order $\leq 2(k+|\alpha|)-1$. Moreover, the model operator of P_0 is $P_{0(2k)} = f^0(0)df^1(0) \wedge \ldots \wedge df^{2k}(0)$, and so we have

$$(P_{0(2k)} G_R)(0,1) = (4\pi)^{-\frac{n}{2}} f^0(0) df^1(0) \wedge \ldots \wedge df^{2k}(0) \wedge \hat{A}(R^M(0)).$$

Applying Proposition 11.7.3 then gives the claim. $\square$

By the Mellin's formula, for $\Re s > 1$ we have

$$|\not{D}_M|^{-2s} = (\not{D}_M^2)^{-s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}(1-\Pi_0)e^{-t\not{D}_M^2}dt, \tag{11.7}$$

where Π_0 is the orthogonal projection onto $\ker \not{D}_M^2 = \ker \not{D}_M$. The above integral converges in $\mathcal{L}\left(L^2(M,\not{S})\right)$ because

$$\|(1-\Pi_0)e^{-t\not{D}_M^2}\| = e^{-\mu t} \qquad \forall t \geq 0, \tag{11.8}$$

where μ is the smallest non-zero eigenvalue of $\not{D}_M^2$.

It is useful to rewrite (11.7) in the form

$$|\not{D}_M|^{-2s} = \Gamma(s)^{-1}\int_0^1 t^{s-1}e^{-t\not{D}_M^2}dt + R(s), \tag{11.9}$$

where we have set

$$\begin{aligned} R(s) :=& -\Gamma(s)^{-1}\int_0^1 t^{s-1}\Pi_0 e^{-t\not{D}_M^2}dt + \Gamma(s)^{-1}\int_1^\infty t^{s-1}(1-\Pi_0)e^{-t\not{D}_M^2}dt \\ =& -s^{-1}\Gamma(s)^{-1}\Pi_0 + \Gamma(s)^{-1}e^{-\frac{1}{4}\not{D}_M^2}\left(\int_{\frac{1}{2}}^\infty (t+\frac{1}{2})^{s-1}(1-\Pi_0)e^{-t\not{D}_M^2}dt\right)e^{-\frac{1}{4}\not{D}_M^2}. \end{aligned} \tag{11.10}$$

The projection Π_0 is a smoothing operator, as is $e^{-\frac{1}{4}\not{D}_M^2}$. Moreover, the function $s^{-1}\Gamma(s)^{-1}$ is entire and using (11.8) we see that $s \rightarrow \int_{\frac{1}{2}}^\infty (t+\frac{1}{2})^{s-1}(1-\Pi_0)e^{-t\not{D}_M^2}dt$ is a holomorphic map from $\mathbb{C}$ to $\mathcal{L}\left(L^2(M,\not{S})\right)$. It then follows that $(R(s))_{s\in\mathbb{C}}$ is a holomorphic family of smoothing operators.

In addition, as the principal symbol of $\not{D}_M^2$ is scalar (it is equal to $\mathrm{id}_{\not{S}_x}|\xi|_g^2$), it commutes with all principal symbols. Therefore, by arguing as in (**??**) we see that if $P \in \Psi^m(M,\not{S})$, then the commutator $[\not{D}_M^2, P]$ is a ΨDO of order $\leq m+1$. An immediate induction then shows that

$$P^{[j]} \in \Psi^{m+j}(M,\not{S}) \qquad \forall j \in \mathbb{N}.$$

In particular, we see that the operator $\not{P}_\alpha = f^0 c(df^1)^{[\alpha_1]}\cdots c(df^{2k})^{[\alpha_{2k}]}$ is a ΨDO of order $\leq \alpha_1 + \cdots + \alpha_{2k} = |\alpha|$.

Bearing all this in mind and using (11.9), we see that, for $\Re z > -(k+\frac{1}{2}|\alpha|)$,

$$\not{P}_\alpha|\not{D}_M|^{-2(k+|\alpha|)-2z} = \Gamma(k+|\alpha|+z)^{-1}\int_0^1 t^{k+|\alpha|+z-1}\not{P}_\alpha e^{-t\not{D}_M^2}dt + \not{P}_\alpha R(k+|\alpha|+z). \tag{11.11}$$

The convergence of the above integral is ensured by the equality

$$\begin{aligned} t^{k+|\alpha|+z-1}\not{P}_\alpha e^{-t\not{D}_M^2} =& \\ & t^{k+|\alpha|+z-1}\not{P}_\alpha\Pi_0 + t^{k+\frac{1}{2}|\alpha|+z-1}\not{P}_\alpha|\not{D}_M|^{-|\alpha|}.(t\not{D}_M^2)^{\frac{|\alpha|}{2}}(1-\Pi_0)e^{-t\not{D}_M^2}, \end{aligned} \tag{11.12}$$

and the following facts:

- The operators $\not{P}_\alpha\Pi_0$ and $\not{P}_\alpha|\not{D}_M|^{-|\alpha|}$ are bounded, since the former is smoothing and the latter is a zeroth order ΨDO.
- For all $t > 0$,

$$\|(t\not{D}_M^2)^{\frac{|\alpha|}{2}}(1-\Pi_0)e^{-t\not{D}_M^2}\| \leq \sup_{\lambda\geq\mu}\lambda^{\frac{|\alpha|}{2}}e^{-\lambda}.$$

Actually, for all $\epsilon > 0$, we can rewrite (11.12) in the form,

$$t^{k+|\alpha|+z-1} \not{P}_\alpha e^{-t\not{D}_M^2} =$$
$$t^{k+|\alpha|+z-1} \not{P}_\alpha \Pi_0 + t^{k+\frac{1}{2}|\alpha|+z-1+n+\epsilon} \not{P}_\alpha |\not{D}_M|^{-|\alpha|-n-\epsilon} . (t\not{D}_M^2)^{\frac{|\alpha|+n+\epsilon}{2}} (1-\Pi_0) e^{-t\not{D}_M^2}.$$

Here both $\not{P}_\alpha \Pi_0$ and $\not{P}_\alpha |\not{D}_M|^{-|\alpha|-n-\epsilon}$ are trace-class operators, since the former is a smoothing operator and the latter is a ΨDO of order $-n-\epsilon < -n$. Therefore, if $\Re z > -(k + \frac{1}{2}|\alpha| + n)$, then the integral in (11.11) actually converges in $\mathcal{L}^1\left(L^2(M, \not{S})\right)$, and hence we may write

$$\text{(11.13)} \quad \operatorname{Str}\left[\not{P}_\alpha |\not{D}_M|^{-2(k+|\alpha|)-2z}\right] =$$
$$\Gamma(k+|\alpha|+z)^{-1} \int_0^1 t^{k+|\alpha|+z-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] dt + \operatorname{Str}\left[\not{P}_\alpha R(k+|\alpha|+z)\right].$$

Observe that, as $\{\not{P}_\alpha R(k+|\alpha|+z)\}_{z\in\mathbb{C}}$ is a holomorphic family of smoothing operators, the function $z \to \operatorname{Str}\left[\not{P}_\alpha R(k+|\alpha|+z)\right]$ is entire. Therefore, using (11.5) and (11.13) we get

$$\text{(11.14)} \quad ⨍ \gamma \not{P}_\alpha |\not{D}_M|^{-2(|\alpha|+k)} = 2 \operatorname{Res}_{z=0} \operatorname{Str}\left[\not{P}_\alpha |\not{D}_M|^{-2(k+|\alpha|)-2z}\right]$$
$$= 2\Gamma(k+|\alpha|)^{-1} \operatorname{Res}_{z=0} \int_0^1 t^{k+|\alpha|+z-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] dt.$$

Furthermore, the potential singularity at $z=0$ of $\int_0^1 t^{k+|\alpha|+z-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] dt$ only depends on the behavior of $\operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right]$ as $t \to 0^+$.

If $\alpha \neq 0$, then by (11.6) we have

$$t^{k+|\alpha|-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] = \mathrm{O}(1),$$

and hence $\int_0^1 t^{k+|\alpha|+z-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] dt$ is regular at $z=0$. Thus,

$$\text{(11.15)} \qquad ⨍ \gamma \not{P}_\alpha |\not{D}_M|^{-2(|\alpha|+k)} = 0 \qquad \text{if } \alpha \neq 0.$$

If $\alpha = 0$, then (11.6) gives

$$t^{k-1} \operatorname{Str}\left[\not{P}_0 e^{-t\not{D}_M^2}\right] = t^{-1}\beta_k + \mathrm{O}(1),$$

where we have set $\beta_k := (2i\pi)^{-\frac{n}{2}} \int_M f^0 df^1 \wedge \ldots \wedge df^{2k} \wedge \hat{A}(R^M)^{(n-2k)}$. Thus,

$$\int_0^1 t^{k+z-1} \operatorname{Str}\left[\not{P}_\alpha e^{-t\not{D}_M^2}\right] dt = \int_0^1 t^{z-1} \beta_k dt + h(z) = z^{-1}\beta_k + h(z),$$

where $h(z)$ is a function which is regular near $z=0$. Combining this with (11.14) then gives

$$\text{(11.16)} \qquad ⨍ \gamma \not{P}_0 |\not{D}_M|^{-2k} = 2\Gamma(k)^{-1}\beta_k.$$

Combining (11.4) with (11.15) and (11.16) we obtain

$$\varphi_{2k}(f^0,\ldots,f^{2k}) = c_{k,0}⨍ \gamma \not{P}_0|\not{D}_M|^{-2k} = \frac{1}{2}\frac{\Gamma(k)}{(2k)!}.2\Gamma(k)^{-1}\beta_k$$
$$= \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!}\int_M f^0 df^1\wedge\cdots\wedge df^{2k}\wedge \hat{A}(R^M)^{(n-2k)}.$$

This proves (11.3) when $k\geq 1$.

It remains to compute φ_0. Let $f^0\in C^\infty(M)$. As $\Gamma(z)\sim z^{-1}$ near $z=0$, we have

$$\begin{aligned}(11.17)\quad \varphi_0(f^0) &= \operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0(|\not{D}_M|^{-z}+\Pi_0)\right]\right\}\\ &= 2\operatorname{Res}_{z=0}\left\{\Gamma(2z)\operatorname{Str}\left[f^0|\not{D}_M|^{-2z}\right]\right\}+\operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0\Pi_0\right]\right\}\\ &= \operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0|\not{D}_M|^{-2z}\right]\right\}+\operatorname{Str}\left[f^0\Pi_0\right].\end{aligned}$$

As in (11.13), for $\Re z>n$ we have

$$(11.18)\quad \Gamma(z)\operatorname{Str}\left[f^0|\not{D}_M|^{-2z}\right] = \int_0^1 t^{z-1}\operatorname{Str}\left[f^0e^{-t\not{D}_M^2}\right]dt+\Gamma(z)\operatorname{Str}\left[f^0R(z)\right].$$

The local index theorem (i.e., Theorem 8.10.5) implies that

$$t^{-1}\operatorname{Str}\left[f^0e^{-t\not{D}_M^2}\right] = t^{-1}\int_M f^0(x)\operatorname{Str}_{\not{S}} k_t(x,x) = t^{-1}\beta_0+\mathrm{O}(1),$$

where we have set $\beta_0 = (2i\pi)^{-\frac{n}{2}}\int_M f^0\hat{A}(R^M)^{(n)}$. Therefore, by using similar arguments as those for proving (11.16) we get

$$\operatorname{Res}_{z=0}\int_0^1 t^{z-1}\operatorname{Str}\left[f^0e^{-t\not{D}_M^2}\right]dt = \beta_0.$$

Furthermore, as (11.10) shows that $R(0)=-\Pi_0$, we have

$$\operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0R(z)\right]\right\} = \operatorname{Str}\left[f^0R(0)\right] = -\operatorname{Str}\left[f^0\Pi_0\right].$$

Therefore, taking residues at $z=0$ of both sides of (11.18) yields

$$(11.19)\quad \operatorname{Res}_{z=0}\left\{\Gamma(z)\operatorname{Str}\left[f^0|\not{D}_M|^{-2z}\right]\right\} = \beta_0-\operatorname{Str}\left[f^0\Pi_0\right].$$

Combining (11.17) with (11.19) gives

$$\varphi_0(f^0) = \beta_0-\operatorname{Str}\left[f^0\Pi_0\right]+\operatorname{Str}\left[f^0\Pi_0\right] = \beta_0 = (2i\pi)^{-\frac{n}{2}}\int_M f^0\hat{A}(R^M)^{(n)}.$$

This is Eq. (11.3) when $k=0$. The proof is thus complete. □

Let $e\in M_q(C^\infty(M))$, $e^2=e^*=e$ and let us denote by E the vector bundle $\operatorname{im} e$; this is a subbundle of the trivial Hermitian bundle $M\times\mathbb{C}^q$. We endow E with the induced Hermitian metric and the Grassmanian connection $\nabla^0 := e(d\otimes 1_q)$.

Let $\not{D}_E : C^\infty(M,\not{S}\otimes E)\rightarrow C^\infty(M,\not{S}\otimes E)$ be the twisted Dirac operator associated to ∇^0, that is,

$$\not{D}_E := \not{D}_M\otimes 1_E+(c\otimes 1_E)(1_{\not{S}}\otimes\nabla^0).$$

As we are using the Grassmanian connection, we know that

$$\not{D}_E = (\not{D}_M)_e = e(\not{D}_M\otimes 1_q).$$

Thus,

$$(11.20)\quad \operatorname{ind}\not{D}_E = \operatorname{ind}(\not{D}_M)_e = \operatorname{ind}_{\not{D}_M}[e] = \langle\varphi_{\mathrm{CM}},e\rangle.$$

Observe that (11.3) exactly means that the CM cocycle φ_{CM} agrees with the even cocycle φ_C associated to the even de Rham current C defined by

$$\langle C,\omega\rangle = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M)\wedge\omega\right]^{(n)} \qquad \forall\omega\in C^\infty(M,\Lambda^{\mathrm{ev}}_{\mathbb{C}}T^*M). \tag{11.21}$$

That is, C is the Poincaré dual of the even form $(2i\pi)^{-\frac{n}{2}}\hat{A}(R^M)$. Therefore, by the results of Chapter 10,

$$\langle\varphi_{\mathrm{CM}}, e\rangle = \langle\varphi_C, e\rangle = \langle C, \mathrm{Ch}(F^0)\rangle,$$

where F^0 is the curvature of ∇^0 and $\mathrm{Ch}(F^0)$ its Chern form. Combining this with (11.20) and (11.21) then gives

$$\mathrm{ind}\,\not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M)\wedge\mathrm{Ch}(F^0)\right]^{(n)}.$$

This is the local index formula of Atiyah-Singer for $\not{D}_E$.

Bibliography

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