1. $\mathbb{Z}_2$-graded spaces and algebras:

Def: A $\mathbb{Z}_2$-graded vector space (or superspace) is a vector space with a $\mathbb{Z}_2$-grading,

$$E = E^+ \oplus E^-$$

If E and F are superspaces, then $E \otimes F$ is a superspace w/ $\mathbb{Z}_2$-grading

$$(E\otimes F) = (E\otimes F)^+ \oplus (E\otimes F)^-$$

$$(E\otimes F)^+ = (E^+ \otimes F^+) \oplus (E^- \otimes F^-)$$

$$(E\otimes F)^- = (E^+ \otimes F^-) \oplus (E^- \otimes F^+)$$

Def: A $\mathbb{Z}_2$-graded algebra (or superalgebra) is an algebra A together w/ a $\mathbb{Z}_2$-grading

$$A = A^+ \oplus A^-$$

in such way that

$$A^+ A^\pm \subset A^\pm \quad \text{and} \quad A^- A^\pm \subset A^\mp.$$

- On a superalgebra A the $\mathbb{Z}_2$-graded commutator (or supercommutator) is defined by

$$[a,b]' = \begin{cases} ab - ba & \text{if } a \text{ or } b \text{ is in } A^+, \\ ab + ba & \text{if both } a \text{ and } b \text{ are in } A^-. \end{cases}$$

- Let E be a superspace. Then $\operatorname{End}(E)$ is a superalgebra with $\mathbb{Z}_2$-grading

$$\operatorname{End}(E) = \operatorname{End}(E)^+ \oplus \operatorname{End}(E)^-$$

$$\operatorname{End}(E)^+ = \{T;\ T(E^\pm) \subset E^\pm\} = \left\{\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix};\ A^\pm \in \operatorname{End}(E^\pm)\right\}$$

$$\operatorname{End}(E)^- = \{T;\ T(E^\pm) \subset E^\mp\} = \left\{\begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix};\ B^\pm \in \mathcal{L}(E^\pm, E^\mp)\right\}$$

Under the splitting $E = E^+ \oplus E^-$ the $\mathbb{Z}_2$-grading operator is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$\operatorname{End}(E)^\pm = \{T;\ \gamma T = \pm T\gamma\}$$

Def: The supertrace of $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{End}(E)$ is

$$\operatorname{Str}_E(T) = \operatorname{Tr}_{E^+}(A) - \operatorname{Tr}_{E^-}(D)$$
$$= \operatorname{Tr}_E(\gamma T).$$

Lemma 1.1: For all $S, T \in \operatorname{End}(E)$,

$$\operatorname{Str}_E [S,T]' = 0.$$

Proof: It is enough to prove this when S and T are in $\operatorname{End}(E)^\pm$.

If S is in $\mathrm{End}(E)^{\pm}$ and $T \in \mathrm{End}(E)^{\mp}$, then both ST, TS, and hence $[S,T]'$, are in $\mathrm{End}(E)^{-}$. Since the supertrace vanishes on $\mathrm{End}(E)^{-}$, we see that, in this case, $\mathrm{Str}_E[S,T]' = 0$.

Suppose that both S and T are in $\mathrm{End}(E)^{+}$. Then $[S,T]' = [S,T]$ and γ commutes with S and T. Thus,

$$\mathrm{Str}_E[S,T]' = \mathrm{Tr}_E\{\gamma[S,T]\} = \mathrm{Tr}_E[\gamma S, T] = 0.$$

It remains to look at the case where both S and T are in $\mathrm{End}(E)^{-}$. Write $S = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix}$. Then

$$[S,T]' = ST + TS = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}\begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix}\begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix} = \begin{pmatrix} A_-B_+ + B_-A_+ & 0 \\ 0 & A_+B_- + B_+A_- \end{pmatrix}$$

Thus,

$$\begin{aligned} \mathrm{Str}_E[S,T] &= \mathrm{Tr}_{E^+}(A_-B_+ + B_-A_+) - \mathrm{Tr}_{E^-}(A_+B_- + B_+A_-) \\ &= \{\mathrm{Tr}_{E^+}(A_-B_+) - \mathrm{Tr}_{E^-}(B_+A_-)\} + \{\mathrm{Tr}_{E^+}(B_-A_+) - \mathrm{Tr}_{E^-}(A_+B_-)\} \\ &= 0. \end{aligned}$$

The proof is complete. □

Let $A = A_+ \oplus A_-$ be a superalgebra.

Def.: A $\mathbb{Z}_2$-graded module over A is a left-module E together with a $\mathbb{Z}_2$-grading,

$$E = E_+ \oplus E_-,$$

such that

$$A^+ . E^{\pm} \subset E^{\pm} \quad \text{and} \quad A^- . E^{\pm} \subset E^{\mp}.$$

Ex: Any superspace E is a $\mathbb{Z}_2$-graded module over the superalgebra $A := \mathrm{End}(E)$.

If E and F are $\mathbb{Z}_2$-graded module over A, then $\mathrm{Hom}_A(E,F)$ is a superspace with $\mathbb{Z}_2$-grading,

$$\mathrm{Hom}_A(E,F) = \mathrm{Hom}_A(E,F)^+ \oplus \mathrm{Hom}_A(E,F)^-$$

where

$$\mathrm{Hom}_A(E,F)^+ = \{T \in \mathrm{Hom}(E,F);\ T(E^{\pm}) \subset F^{\pm}\}$$
$$\mathrm{Hom}_A(E,F)^- = \{T \in \mathrm{Hom}(E,F);\ T(E^{\pm}) \subset F^{\mp}\}$$

2. The Exterior Algebra $\Lambda^* \mathbb{R}^n$:

Let $\Lambda^* \mathbb{R}^n = \bigoplus_{j=0}^{n} \Lambda^j \mathbb{R}^n$ be the exterior algebra of $\mathbb{R}^n$. (In the sequel we shall freely identify $\Lambda^1 \mathbb{R}^n$ with $\mathbb{R}^n$).

The exterior algebra $\Lambda^* \mathbb{R}^n$ is a superalgebra with $\mathbb{Z}_2$-grading

$$\Lambda^* \mathbb{R}^n = \Lambda^{ev} \mathbb{R}^n \oplus \Lambda^{odd} \mathbb{R}^n, \text{ where}$$

$$\Lambda^{ev} \mathbb{R}^n = \bigoplus_{j \text{ even}} \Lambda^j \mathbb{R}^n \quad \text{and} \quad \Lambda^{odd} \mathbb{R}^n = \bigoplus_{j \text{ odd}} \Lambda^j \mathbb{R}^n.$$

Let $v \in \mathbb{R}^n$. We can associate to v two endomorphisms of $\Lambda^* \mathbb{R}^n$ as follows.

. The first operator is the <u>exterior multiplication</u> by v:

$$\varepsilon(v)\eta := v \wedge \eta \qquad \forall \eta \in \Lambda^* \mathbb{R}^n.$$

. The 2nd operator is the <u>interior multiplication</u> by v:

$$i(v)(\omega^1 \wedge \cdots \wedge \omega^l) = \sum_{1 \leq j \leq l} (-1)^{j-1} \langle v, \omega^j \rangle \, \omega^1 \wedge \cdots \wedge \underbrace{\hat{\omega}^j}_{\omega^j \text{ missing}} \wedge \cdots \wedge \omega^l.$$

Notice that $\varepsilon(v)$ raises the degree of forms, while $i(v)$ decreases it.

Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ (e.g., the canonical basis). Let $\mathcal{J}$ be the family of ordered subsets $I = (i_1, \ldots, i_l)$, $i_1 < \cdots < i_l$, including the empty set. For $I \in \mathcal{J}$ we set

$$e^I = \begin{cases} e^{i_1} \wedge \cdots \wedge e^{i_l} & \text{if } I = (i_1, \ldots, i_l), \\ 1 & \text{if } I = \emptyset. \end{cases}$$

Then $\{e^I\}_{I \in \mathcal{J}}$ is an orthonormal basis of $\Lambda^* \mathbb{R}^n$.

In the sequel, for $I = (i_1, \ldots, i_l) \in \mathcal{J} \setminus \{\emptyset\}$ and $j \in \{1, \ldots, n\} \setminus J$ we define

$$\varepsilon(j, J) := \sup \{q;\ i_q < j\}.$$

Therefore if $p = \varepsilon(j, J)$ then $I \cup \{j\} = (i_1, \ldots, i_p, j, i_{p+1}, \ldots, i_l)$. If $J = \emptyset$ we set $\varepsilon(j, \emptyset) = 1$.

<u>Lemma 2.1.</u> Let $j \in \{1, \ldots, n\}$. Then, for all $I \in \mathcal{J}$,

$$\varepsilon(e^j) e^I = \begin{cases} 0 & \text{if } j \in I, \\ \varepsilon(j, I)\, e^{I \cup \{j\}} & \text{if } j \notin I, \end{cases}$$

and

$$i(e^j) e^I = \begin{cases} \varepsilon(j, I \setminus \{j\})\, e^{I \setminus \{j\}} & \text{if } j \in I, \\ 0 & \text{if } j \notin I. \end{cases}$$

Using these formulas it is not difficult to check that

$$\varepsilon(e^j)\varepsilon(e^k) + \varepsilon(e^k)\varepsilon(e^j) = i(e^j) i(e^k) + i(e^k) i(e^j) = 0$$

$$\varepsilon(e^j) i(e^k) + i(e^k) \varepsilon(e^j) = \delta^{jk} = \langle e^j, e^k \rangle.$$

Therefore, by bilinearity we obtain:

Lemma 2.2: For all $v, w \in \mathbb{R}^n$,
$$\varepsilon(v)\varepsilon(w) + \varepsilon(w)\varepsilon(v) = i(v)i(w) + i(w)i(v) = 0,$$
$$\varepsilon(v)i(w) + i(w)\varepsilon(v) = \langle v, w \rangle.$$

Observe also that the formulas of Lemma 2.1 show that, with respect to the orthonormal basis $\{e^I\}_{I \in \mathcal{J}}$, the matrices of the endomorphisms $\varepsilon(e^j)$ and $i(e^j)$ are transposes of each other. Thus,
$$\varepsilon(e^j)^* = i(e^j).$$
By linearity this gives:

Lemma 2.3: For all $v \in \mathbb{R}^n$,
$$\varepsilon(v)^* = i(v) \text{ and } i(v)^* = \varepsilon(v).$$

3. The Clifford algebra $Cl(\mathbb{R}^n)$:

Def.: The Clifford algebra of $\mathbb{R}^n$, denoted $Cl(\mathbb{R}^n)$, is the subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet \mathbb{R}^n)$ generated the operators,
$$c(v) := \varepsilon(v) - i(v), \quad v \in \mathbb{R}^n.$$
Rmk: The operator $c(v)$ is called the Clifford multiplication by v.

Using Lemma 2.2 and Lemma 2.3 we get

Lemma 3.1: (i) For all $v, w \in \mathbb{R}^n$,
$$c(v)c(w) + c(w)c(v) = -2\langle v, w \rangle.$$
(ii) For all $v \in \mathbb{R}^n$,
$$c(v)^* = -c(v).$$
It follows from (i) that for any orthonormal basis $\{e^1, \ldots, e^n\}$ we have
$$c(e^j)c(e^k) + c(e^k)c(e^j) = -2\delta^{jk}.$$

This implies that, if $j \neq k$,
$$c(e^j)c(e^k) = -c(e^k)c(e^j),$$
and
$$c(e^j)^2 = -1.$$

Moreover, (ii) implies that $Cl(\mathbb{R}^n)$ is closed under taking adjoints, i.e., $Cl(\mathbb{R}^n)$ is an involutive sub-algebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet \mathbb{R}^n)$ (and so this is a C^*-algebra over $\mathbb{R}$).

Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. As $\{e^I\}_{I \in \mathcal{J}}$ is an orthonormal basis of $\Lambda \mathbb{R}^n$, the linear map $\mathbb{R}^n \ni v \longrightarrow c(v) \in Cl(\mathbb{R}^n)$ is uniquely extended into the linear map
$$c : \Lambda^\bullet \mathbb{R}^n \longrightarrow Cl(\mathbb{R}^n)$$

such that

$$c(e^I) = \begin{cases} c(e^{i_1})\cdots c(e^{i_l}) & \text{if } I=(i_1,\dots,i_l), \\ 1 & \text{if } I=\phi. \end{cases}$$

On the other, as $Cl(\mathbb{R}^n)$ is an algebra of endomorphisms of $\Lambda^\bullet\mathbb{R}^n$ we have a natural linear map $\sigma: Cl(\mathbb{R}^n) \longrightarrow \Lambda^\bullet\mathbb{R}^n$ defined by

$$\sigma(a) = a.1 \quad \forall a \in Cl(\mathbb{R}^n).$$

<u>Prop. 3.2:</u> The linear maps c and σ are inverses of each other.

<u>Proof:</u> It is clear that c is onto because $1, c(e^1), \dots, c(e^n)$ generate $Cl(\mathbb{R}^n)$.
Furthermore, let $I \in \mathcal{S}$. If $I=\phi$, then $c(e^I)=1$ and hence $c(e^I).1 = 1$. Suppose that $I=(i_1,\dots,i_l)\in\mathcal{S}\setminus\phi$. As $c(e^{i_p}) = \varepsilon(e^{i_p}) - \iota(e^{i_p})$ and $\iota(e^{i_p})$ lowers the degree, we see that

$$c(e^I) = c(e^{i_1})\cdots c(e^{i_l}) = \left(\varepsilon(e^{i_1}) - \iota(e^{i_1})\right)\cdots\left(\varepsilon(e^{i_l}) - \iota(e^{i_l})\right)$$
$$= \varepsilon(e^{i_1})\cdots\varepsilon(e^{i_l}) + T,$$

where T is an operator which lowers the degree. In particular, $T.1 = 0$, and so we have

$$c(e^I).1 = \varepsilon(e^{i_1})\cdots\varepsilon(e^{i_l}).1 = e^{i_1}\wedge\cdots\wedge e^{i_l} = e^I.$$

This shows that $\sigma\circ c(e^I) = e^I \ \forall I\in\mathcal{S}$, and hence

$$\sigma\circ c = \mathrm{id}.$$

This proves that c is injective and τ is a left-inverse for c. As we already know that c is onto, it follows that c is a linear isomorphism with inverse τ. The proof is complete. $\square$

<u>Rmk.:</u> An immediate consequence of the fact that σ is the inverse of c, is the fact that the map c <u>does not</u> depend on the choice of the orthonormal basis $\{e^i\}$. In fact, for any orthogonal basis $\{f^i\}$, we have

$$c(f^{i_1}\wedge\cdots\wedge f^{i_l}) = c(f^{i_1})\cdots c(f^{i_l}), \quad \forall (i_1,\dots,i_l)\in\mathcal{S}\setminus\phi.$$

It should be stressed that c and σ are not isomorphisms of algebras, because we have

<u>Lemma 3.3:</u> For all $v, w\in\mathbb{R}^n$,

$$c(v\wedge w) = c(v)c(w) + \langle v, w\rangle.$$

<u>Proof:</u> Let $\{e^1,\dots,e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. If $j\neq l$, then

$$c(e^j\wedge e^l) = c(e^j)c(e^l) = c(e^j)c(e^l) + \langle e^j, e^l\rangle.$$

Furthermore, as $c(e^j)^2=-1$, we see that
$$0=c(e^j\wedge e^j)=c(e^j)c(e^j)+\langle e^j,e^j\rangle.$$
Thus the lemma holds when v and w are elements of the basis $\{e^j\}$. By bilinearity it holds for all v and w in $\mathbb{R}^n$. The proof is complete. □

In general, we have

<u>Lemma 3.4</u>: Let $\xi\in\Lambda^k\mathbb{R}^n$ and $\eta\in\Lambda^l\mathbb{R}^n$ with $k+l\leq n$. Then
$$c(\xi\wedge\eta)=c(\xi)c(\eta) \quad \text{mod } c(\Lambda^{k+l-2}\mathbb{R}^n).$$

<u>Proof</u>: Let $\{e^1,\ldots,e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. It is enough to prove the lemma when $\xi=e^I$ and $\eta=e^J$ for some $I=(i_1,\ldots,i_k)$ and $J=(j_1,\ldots,j_l)$ in $\mathcal{J}$.

Suppose that $I\cap J\neq\emptyset$. Then $e^I\wedge e^J=0$, so we need to show that $c(e^I)c(e^J)$ is contained in $c(\Lambda^{k+l-2}\mathbb{R}^n)$. Let $j\in I\cap J$. As $c(e^j)c(e^k)=-c(e^k)c(e^j)$ we see that
$$c(e^I)=c(e^{i_1})\cdots c(e^{i_k})=\pm c(e^{I\setminus\{j\}})c(e^j),$$
$$c(e^J)=c(e^{j_1})\cdots c(e^{j_l})=\pm c(e^j)c(e^{J\setminus\{j\}}),$$
Combining this with the equality $c(e^j)^2=-1$, then gives
$$c(e^I)c(e^J)=\pm c(e^{I\setminus\{j\}})c(e^j)c(e^j)c(e^{J\setminus\{j\}})$$
$$=\pm c(e^{I\setminus\{j\}})c(e^{J\setminus\{j\}})\in c(\Lambda^{k+l-2}\mathbb{R}^n).$$

Assume now that $I\cap J=\emptyset$. Write $\{1,\ldots,n\}\setminus(I\cup J)=\{m_1,\ldots,m_{n-(k+l)}\}$. As $\{e^{i_1},\ldots,e^{i_k},e^{j_1},\ldots,e^{j_l},e^{m_1},\ldots,e^{m_{n-(k+l)}}\}$ is an orthonormal basis of $\mathbb{R}^n$,
$$c(e^I\wedge e^J)=c(e^{i_1}\wedge\cdots\wedge e^{i_k}\wedge e^{j_1}\wedge\cdots\wedge e^{j_l})=c(e^{i_1})\cdots c(e^{i_k})c(e^{j_1})\cdots c(e^{j_l})=c(e^I)c(e^J).$$
This completes the proof of the lemma. □

Next, the $\mathbb{Z}_2$-grading of $\Lambda^\bullet\mathbb{R}^n$ gives rise to $\mathbb{Z}_2$-grading on $Cl(\mathbb{R}^n)$. Namely,
$$Cl(\mathbb{R}^n)=Cl^+(\mathbb{R}^n)\oplus Cl^-(\mathbb{R}^n),$$
$$Cl^+(\mathbb{R}^n)=c(\Lambda^{ev}\mathbb{R}^n) \quad\text{and}\quad Cl^-(\mathbb{R}^n)=c(\Lambda^{odd}\mathbb{R}^n)$$

<u>Prop. 3.5</u>: $Cl(\mathbb{R}^n)$ is a superalgebra.

<u>Proof</u>: For $v\in\mathbb{R}^n$ the endomorphisms $\varepsilon(v)$ and $\iota(v)$ shift the degree by 1, so these are elements of $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^-$, and hence the Clifford multiplication $c(v)$ is in $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^-$ too. As a product of an even (resp., odd) number of elements of $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^-$ is contained in $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^+$ (resp., $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^-$), we see that $Cl^+(\mathbb{R}^n)$ (resp., $Cl^-(\mathbb{R}^n)$) is contained in $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^+$ (resp., $\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^-$), and hence
$$Cl^\pm(\mathbb{R}^n)=\mathrm{End}(\Lambda^\bullet\mathbb{R}^n)^\pm\cap Cl(\mathbb{R}^n).$$

Thus,

$$Cl^+(\mathbb{R}^n) \cdot Cl^\pm(\mathbb{R}^n) \subset \left(\mathrm{End}(\Lambda^\bullet \mathbb{R}^n)^+ \cdot \mathrm{End}(\Lambda^\bullet \mathbb{R}^n)^\pm\right) \cap Cl(\mathbb{R}^n) \subset Cl^\pm(\mathbb{R}^n),$$
$$Cl^-(\mathbb{R}^n) \cdot Cl^\pm(\mathbb{R}^n) \subset \left(\mathrm{End}(\Lambda^\bullet \mathbb{R}^n)^- \cdot \mathrm{End}(\Lambda^\bullet \mathbb{R}^n)^\pm\right) \cap Cl(\mathbb{R}^n) \subset Cl^\mp(\mathbb{R}^n).$$

This proves that $Cl(\mathbb{R}^n)$ is a superalgebra. $\square$

As $Cl(\mathbb{R}^n)$ is a superalgebra, we can look at supercommutators. The following result, although seemingly simple, will has many important consequences in the sequel.

Lemma 3.6. Let $\omega \in \mathbb{R}^n$ and $\eta \in \Lambda^\bullet \mathbb{R}^n$. Then
$$[c(\omega), c(\eta)]' = -2c(\iota(\omega)\eta).$$

Proof: Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then it is enough to prove the lemma when $\eta = e^I$ and $\omega = e^j$ for some $I = (i_1, \ldots, i_\ell) \in \mathcal{I}$ and $j \in \{1, \ldots, n\}$. Furthermore, upon reordering the basis $\{e^i\}$ we may assume that $j = 1$ and either $I = (1, \ldots, \ell)$ if $j \in I$, or $I = (2, \ldots, \ell+1)$ otherwise.

Suppose that $j \notin I$. Then $\iota(e^j)e^I = 0$. Moreover, as $c(e^p)c(e^q) = -c(e^q)c(e^p)$ for $p \neq q$, we have
$$c(e^j)c(e^I) = c(e^1)c(e^2)\cdots c(e^\ell) = (-1)^\ell c(e^2)\cdots c(e^\ell)c(e^1) = (-1)^\ell c(e^j)c(e^I).$$

Thus,
$$[c(e^j), c(e^I)]' = c(e^j)c(e^I) - (-1)^\ell c(e^I)c(e^j) = 0 = -2c(\iota(e^j)e^I).$$

Assume now that $j \in I$. Then
$$\iota(e^j)e^I = \iota(e^1)(e^1 \wedge \cdots \wedge e^\ell) = e^2 \wedge \cdots \wedge e^\ell.$$

Furthermore, as $c(e^1)^2 = -1$, we have
$$c(e^j)c(e^I) = c(e^1)c(e^1)c(e^2)\cdots c(e^\ell) = -c(e^2)\cdots c(e^\ell),$$
$$c(e^I)c(e^j) = c(e^1)\cdots c(e^\ell)c(e^1) = (-1)^{\ell-1}c(e^1)^2 c(e^2)\cdots c(e^\ell) = (-1)^\ell c(e^2)\cdots c(e^\ell).$$

Thus,
$$\begin{aligned}
[c(e^j), c(e^I)]' &= c(e^j)c(e^I) - (-1)^\ell c(e^I)c(e^j) \\
&= -c(e^2)\cdots c(e^\ell) - (-1)^\ell(-1)^\ell c(e^2)\cdots c(e^\ell) \\
&= -2c(e^2)\cdots c(e^\ell) \\
&= -2c(\iota(e^j)e^I).
\end{aligned}$$

This completes the proof. $\square$

Observe that if $\omega, \omega^1, \ldots, \omega^\ell$ are orthogonal unit vectors, then
$$\omega^1 \wedge \cdots \wedge \omega^\ell = \iota(\omega)(\omega \wedge \omega^1 \wedge \cdots \wedge \omega^\ell).$$

It follows from this that any element of $\Lambda^k \mathbb{R}^n$ w/ $k\leq n-1$ is a linear combination of elements of the form $i(v)\eta$ with $v\in\mathbb{R}^n$ and $\eta\in\Lambda^{k+1}\mathbb{R}^n$. Combining this observation with the previous lemma we obtain: (8)

<u>Lemma 3.7</u>: The supercommutator space of $Cl(\mathbb{R}^n)$ is contained in $c\left(\bigoplus_{j=0}^{n-1}\Lambda^j\mathbb{R}^n\right)$.

As $Cl^+(\mathbb{R}^n)Cl^+(\mathbb{R}^n)\subset Cl^+(\mathbb{R}^n)$ we see that $Cl^+(\mathbb{R}^n)$ is a subalgebra of $Cl(\mathbb{R}^n)$. In fact, we have

<u>Lemma 3.8</u>: If $n\geq 3$, there is an isomorphism of algebras,

$$Cl^+(\mathbb{R}^n)\simeq Cl(\mathbb{R}^{n-1}).$$

<u>Proof</u>: Let $\{e^1,\ldots,e^n\}$ be the canonical basis of $\mathbb{R}^n$. Upon identifying $\mathbb{R}^{n-1}$ with the span of $e^1,\ldots,e^{n-1}$ in $\mathbb{R}^n$ we can identify $Cl(\mathbb{R}^{n-1})$ with the subalgebra of $Cl(\mathbb{R}^n)$ generated by $c(e^1),\ldots,c(e^{n-1})$. Then

$$Cl(\mathbb{R}^n)=Cl(\mathbb{R}^{n-1})\oplus\left(Cl(\mathbb{R}^{n-1})c(e^n)\right)$$

$$Cl^+(\mathbb{R}^n)=Cl^+(\mathbb{R}^{n-1})\oplus\left(Cl^-(\mathbb{R}^{n-1})c(e^n)\right).$$

We also observe that, as $c(e^n)$ anticommutes with the $c(e^j)$, $1\leq j\leq n-1$, we have

$$c(e^n)a=\pm c(e^n)c(a)\qquad\forall a\in Cl^\pm(\mathbb{R}^{n-1})$$

As $Cl(\mathbb{R}^{n-1})=Cl^+(\mathbb{R}^n)\oplus Cl^-(\mathbb{R}^{n-1})$ any $a\in Cl(\mathbb{R}^{n-1})$ can be uniquely written as

$$a=a^++a^-,\qquad a^\pm\in Cl^\pm(\mathbb{R}^{n-1}).$$

We then define a linear map $\alpha: Cl(\mathbb{R}^{n-1})\longrightarrow Cl^+(\mathbb{R}^n)$ by

$$\alpha(a):=a^++a^-c(e^n).$$

This map is onto since $Cl^+(\mathbb{R}^n)=Cl^+(\mathbb{R}^{n-1})\oplus Cl^-(\mathbb{R}^{n-1})c(e^n)$. Moreover, if $\alpha(a)=0$, then $a^+=a^-c(e^n)=0$, and hence $0=a^-\underbrace{c(e^n)c(e^n)}_{=-1}=-a^-$, so that $a=0$. Therefore, α is a linear isomorphism.

It remains to show that α is a morphism of algebras. Let $a,b\in Cl(\mathbb{R}^n)$. Then

$$\alpha(a)\alpha(b)=(a^++a^-c(e^n))(b^++b^-c(e^n))$$
$$=a^+b^++a^-c(e^n)b^-c(e^n)+(a^+b^-c(e^n)+a^-c(e^n)b^+)$$

As $c(e^n)^2=-1$ and $c(e^n)$ commutes (resp. anticommutes) with the elements of $Cl^+(\mathbb{R}^n)$ (resp., $Cl^-(\mathbb{R}^n)$) we see that

$$a^-c(e^n)b^-c(e^n)=-a^-c(e^n)^2b^-=a^-b^-,$$
$$a^-c(e^n)b^+=a^-b^+c(e^n)$$

Therefore,

$$\begin{aligned}\alpha(a)\alpha(b) &= (a^+b^+ + a^-b^-) + (a^+b^- + a^-b^+)c(e^n)\\ &= (ab)^+ + (ab)^- c(e^n)\\ &= \alpha(ab).\end{aligned}$$

This proves that α is a morphism of algebras and completes the proof. □

Next, it is convenient to complexify $Cl(\mathbb{R}^n)$. We introduce

$$Cl_{\mathbb{C}}(\mathbb{R}^n) = Cl(\mathbb{R}^n)\otimes_{\mathbb{R}}\mathbb{C} \quad \text{and} \quad Cl_{\mathbb{C}}^{\pm}(\mathbb{R}^n) = Cl^{\pm}(\mathbb{R}^n)\otimes_{\mathbb{R}}\mathbb{C}.$$

Upon identify $\mathbb{R}^n\otimes_{\mathbb{R}}\mathbb{C}$ with $\mathbb{C}^n$ and $(\Lambda^*\mathbb{R}^n)\otimes_{\mathbb{R}}\mathbb{C}$ with $\Lambda^*\mathbb{C}^n$ we extend the $\mathbb{R}$-linear isomorphisms c and σ into $\mathbb{C}$-linear isomorphisms

$$\Lambda^*\mathbb{C}^n \underset{\sigma}{\overset{c}{\rightleftarrows}} Cl_{\mathbb{C}}(\mathbb{R}^n)$$

We observe that, for all $v, w \in \mathbb{C}^n$,

$$(1) \quad c(v)c(w) + c(w)c(v) = -2\langle \bar{v}, w\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard Hermitian inner product of $\mathbb{C}^n$:

$$\langle v, w\rangle := \sum \bar{v}_j w_j \qquad \forall v=(v_j)\in\mathbb{C}^n \quad \forall w=(w_j)\in\mathbb{C}^n.$$

If $\{e^1,\ldots,e^n\}$ is any <u>oriented</u> orthonormal basis, the volume element of $\mathbb{R}^n$ is

$$\omega_{\mathbb{R}^n} = e^1\wedge\cdots\wedge e^n \in \Lambda^n\mathbb{R}^n.$$

<u>Def.</u>: The <u>chirality operator</u> is the element of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ defined by

$$\begin{aligned}\Gamma &= (i)^{[\frac{n+1}{2}]}c(\omega_{\mathbb{R}^n})\\ &= (i)^{[\frac{n+1}{2}]}c(e^1)\cdots c(e^n),\end{aligned}$$

where $\{e^1,\ldots,e^n\}$ is any oriented orthonormal basis of $\mathbb{R}^n$.

<u>Lemma 39</u>: (i) $\Gamma^2 = 1$.

(ii) $c(v)\Gamma = (-1)^{n-1}\Gamma c(v) \quad \forall v\in\mathbb{R}^n$. (iii) $\Gamma^* = \Gamma$.

<u>Proof.</u> Let $\{e^1,\ldots,e^n\}$ be an oriented orthonormal basis of $\mathbb{R}^n$. Then

$$\begin{aligned}c(\omega_{\mathbb{R}^n})^2 = c(\omega_{\mathbb{R}^n})\cdot c(\omega_{\mathbb{R}^n}) &= c(e^1)\cdots c(e^n)c(e^1)\cdots c(e^n)\\ &= (-1)^{n-1}c(e^1)c(e^1)c(e^2)\cdots c(e^n)c(e^2)\cdots c(e^n)\\ &= (-1)^{\frac{(n-1)n}{2}}c(e^1)c(e^1).c(e^2)c(e^2)\cdots c(e^n)c(e^n)\\ &= (-1)^{\frac{(n-1)n}{2}} -1.-1.\cdots.-1 = (-1)^{\frac{(n-1)n}{2}}(-1)^n = (-1)^{\frac{(n+1)n}{2}}.\end{aligned}$$

If n is even, then $\Gamma = i^{\frac{n}{2}} c(\omega_{\mathbb{R}^n})$, and hence

$$\Gamma^2 = (-1)^{\frac{n}{2}} c(\omega_{\mathbb{R}^n})^2 = (-1)^{\frac{n}{2}} \cdot (-1)^{\frac{(n+1)n}{2}} = (-1)^{\frac{(n+2)n}{2}} = 1,$$

since both n and $n+2$ are even numbers.

If n is odd, then $\Gamma = i^{\frac{n+1}{2}} c(\omega_{\mathbb{R}^n})$, and hence

$$\Gamma^2 = (-1)^{\frac{n+1}{2}} c(\omega_{\mathbb{R}^n}) = (-1)^{\frac{n+1}{2}} (-1)^{\frac{(n+1)n}{2}} = (-1)^{\frac{(n+1)^2}{2}} = 1,$$

since $(n+1)$ is even and so $(n+1)^2$ is a multiple of 4.

In any case we see that $\Gamma^2 = 1$, so the first part of the lemma is proved.

In order to prove the 2nd part, it is enough to do it when $v = e^j$ for some $j \in \{1, \ldots, n\}$. In fact, upon reordering the basis $\{e^1, \ldots, e^n\}$ we may assume that $j = 1$. In that case this is a by product of the proof of the 1st part. Indeed, as we have shown above,

$$c(\omega_{\mathbb{R}^n}) c(e^1) = c(e^1) \cdots c(e^n) c(e^1) = (-1)^{n-1} c(e^1) c(e^1) \cdots c(e^n) = (-1)^{n-1} c(e^1) c(\omega_{\mathbb{R}^n}),$$

and hence $\Gamma c(e^1) = (-1)^{n-1} c(e^1) \Gamma$.

Let us now compute Γ^*. As by Lemma 3.1 $c(e^j)^* = -c(e^j)$, set

$$\Gamma^* = (\bar{i})^{[\frac{n+1}{2}]} c(e^n)^* \cdots c(e^1)^* = i^{[\frac{n+1}{2}]} (-1)^{[\frac{n+1}{2}]+n} c(e^n) \cdots c(e^1) = i^{[\frac{n+1}{2}]} (-1)^{[\frac{n+1}{2}]+n+\frac{n(n-1)}{2}} c(e^1) \cdots c(e^n)$$
$$= (-1)^{[\frac{n+1}{2}]+\frac{n(n+1)}{2}} \Gamma.$$

If n is even, then $[\frac{n+1}{2}] = \frac{n}{2}$, and hence $(-1)^{[\frac{n+1}{2}]+\frac{n(n+1)}{2}} = (-1)^{\frac{n}{2}+\frac{n(n+1)}{2}} = (-1)^{\frac{n(n+2)}{2}} = 1$. If n is odd then $[\frac{n+1}{2}] = \frac{n+1}{2}$, and hence $(-1)^{[\frac{n+1}{2}]+\frac{n(n+1)}{2}} = (-1)^{\frac{n+1}{2}+\frac{n(n+1)}{2}} = (-1)^{\frac{(n+1)^2}{2}} = 1$. In both cases we see that $\Gamma^* = \Gamma$.
The proof is complete. □

Lemma 3.10: Assume n odd. Then Γ is a central element of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ and we have

$$Cl_{\mathbb{C}}^-(\mathbb{R}^n) = \Gamma Cl_{\mathbb{C}}^+(\mathbb{R}^n) \quad \text{and} \quad Cl_{\mathbb{C}}(\mathbb{R}^n) = Cl_{\mathbb{C}}^+(\mathbb{R}^n) \oplus \Gamma Cl_{\mathbb{C}}^+(\mathbb{R}^n).$$

4. The Spinor Representation:

Thm. 4.1 (Spinor Representation, n even): Assume n even. Then:

1. Up to equivalence, the algebra $Cl_{\mathbb{C}}(\mathbb{R}^n)$ has a unique irreducible representation, namely, the spinor representation,

$$\rho : Cl_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow End(\$),$$

where $\$ = \$(\mathbb{R}^n)$ is the space of spinors of $\mathbb{R}^n$ (see definition on page).

2. The representation ρ gives rise to an isomorphism of algebras,

$$Cl_{\mathbb{C}}(\mathbb{R}^n) \simeq End(\$).$$

Rmk.: The unicity up to equivalence means that if $\pi : Cl_{\mathbb{C}}(\mathbb{R}^n) \to End(E)$ is another irreducible representation then $\$$ and E are isomorphic $Cl_{\mathbb{C}}(\mathbb{R}^n)$-modules, i.e., there exists a linear isomorphism $\varphi : \$ \to E$ s.t.

$$\varphi(\rho(a)\sigma) = \pi(a)\varphi(\sigma) \quad \forall \sigma \in \$ \quad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

Rmk.: As we shall $\$$ has dimension $2^{n/2}$, so we get the following isomorphism of algebras,

$$Cl_{\mathbb{C}}(\mathbb{R}^n) \simeq End(\$) \simeq M_{2^{\frac{n}{2}}}(\mathbb{R}).$$

Since $M_{2^{\frac{n}{2}}}(\mathbb{R})$ is a simple algebra, it is clear that $Cl_{\mathbb{C}}(\mathbb{R}^n)$ has a unique irreducible representation.

The next 10 pages are devoted to proving Theorem 4.1. PLEASE TURN OVER.

Proof of Theorem 4.1:

A Construction of the spinor representation:

Set $m = \frac{n}{2}$ and let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$.
$$z^j = \frac{1}{\sqrt{2}}\left(e^{2j} + i e^{2j+1}\right) \quad \text{and} \quad z^{\bar{j}} = \overline{z^j} = \frac{1}{\sqrt{2}}\left(e^{2j} - i e^{2j+1}\right).$$
Then $\{z^j, z^{\bar{j}}\}_{1 \le j \le m}$ is an orthonormal basis of $\mathbb{C}^n$ and we get the orthogonal splitting,
$$\mathbb{C}^n = \Lambda^{1,0}\mathbb{C}^n \oplus \Lambda^{0,1}\mathbb{C}^n,$$
where
$$\Lambda^{1,0}\mathbb{C}^n = \mathrm{Span}\{z^1, \dots, z^m\} \quad \text{and} \quad \Lambda^{1,0}\mathbb{C}^n = \mathrm{Span}\{z^{\bar{1}}, \dots, z^{\bar{m}}\} = \overline{\Lambda^{0,1}\mathbb{C}^n}.$$
This splitting then gives rise to the orthogonal decomposition
$$\Lambda^* \mathbb{C}^n = \bigoplus_{p,q=0}^{m} \Lambda^{p,q}\mathbb{C}^n,$$
where
$$\Lambda^{p,q}\mathbb{C}^n = \left(\Lambda^{1,0}\mathbb{C}^n\right)^p \wedge \left(\Lambda^{0,1}\mathbb{C}^n\right)^q$$
$$= \mathrm{Span}\left\{ z^{i_1} \wedge \dots \wedge z^{i_p} \wedge z^{\bar{j}_1} \wedge \dots \wedge z^{\bar{j}_q} \right\}_{\substack{1 \le i_1 < \dots < i_p \le m \\ 1 \le j_1 < \dots < j_q \le m}}.$$

Furthermore, we see that the algebra $Cl_{\mathbb{C}}(\mathbb{R}^n)$ is generated by the $c(z^j)$'s and the $c(z^{\bar{j}})$'s. Moreover, by using the formula (1) on page 9 we see that
$$c(z^j)c(z^k) + c(z^k)c(z^j) = c(z^{\bar{j}})c(z^{\bar{k}}) + c(z^{\bar{k}})c(z^{\bar{j}}) = 0,$$
$$c(z^j)c(z^{\bar{k}}) + c(z^{\bar{k}})c(z^j) = -2\delta^{jk}, \quad 1 \le j, k \le m \text{ (See Remark page 5)}$$
The spinor bundle is defined to be
$$\$_n := \Lambda^{0,*}\mathbb{C}^n = \bigoplus_{q=0}^{m} \Lambda^{0,q}\mathbb{C}^n$$
We define a linear map ρ from $c(\Lambda^1\mathbb{C}^n) = c(\Lambda^{1,0}\mathbb{C}^n) \oplus c(\Lambda^{0,1}\mathbb{C}^n)$ as follows. If $\bar{z} \in \Lambda^{0,1}\mathbb{C}^n$, then
$$\rho(c(\bar{z}))\sigma = \sqrt{2}\,\varepsilon(\bar{z}),$$
where $\varepsilon(\bar{z})$ is the exterior product by $\bar{z}$, i.e.,
$$\varepsilon(\bar{z})\sigma := \bar{z} \wedge \sigma \quad \forall \sigma \in \Lambda^{0,*}\mathbb{C}^n.$$
If $z \in \Lambda^{1,0}\mathbb{C}^n$, then
$$\rho(c(z)) = -\sqrt{2}\, i(z),$$
where $i(z)$ is the interior product by z, that is,
$$i(z)(\omega^{\bar{1}} \wedge \dots \wedge \omega^{\bar{q}}) = \sum_{j=1}^{q} (-1)^{j-1} \langle \bar{z}, \omega \rangle\, \omega^{\bar{1}} \wedge \dots \wedge \widehat{\omega^{\bar{j}}} \wedge \dots \wedge \omega^{\bar{q}} \quad \forall \omega^{\bar{j}} \in \Lambda^{0,1}\mathbb{C}^n.$$

By using Lemma 2.2 it is not difficult to check that, for $j, l = 1, \ldots, n$,

$$i(z^j)i(z^l) + i(z^l)i(z^j) = \varepsilon(z^{\bar{j}})\varepsilon(z^{\bar{l}}) + \varepsilon(z^{\bar{l}})\varepsilon(z^{\bar{j}}) = 0$$

$$i(z^j)\varepsilon(z^{\bar{l}}) + \varepsilon(z^{\bar{l}})i(z^j) = \delta^{jl},$$

and hence

$$\rho(c(z^j))\rho(c(z^l)) + \rho(c(z^l))\rho(c(z^j)) = 0$$

$$\rho(c(z^{\bar{j}}))\rho(c(z^{\bar{l}})) + \rho(c(z^{\bar{l}}))\rho(c(z^{\bar{j}})) = 0$$

$$\rho(c(z^j))\rho(c(z^{\bar{l}})) + \rho(c(z^{\bar{l}}))\rho(c(z^j)) = -2\delta^{jl}$$

By bilinearity it then follows that, for all $v, w \in \mathbb{C}^n$,

$$\rho(c(v))\rho(c(w)) + \rho(c(w))\rho(c(v)) = -2\langle \bar{v}, w\rangle.$$

<u>Lemma 4.2</u>: Let A be a (unital) algebra over $\mathbb{C}$ and let $\pi : c(\mathbb{C}^n) \to A$ be a $\mathbb{C}$-linear map s.t.

$$(4.1) \quad \pi(c(v))\pi(c(w)) + \pi(c(w))\pi(c(v)) = -2\langle \bar{v}, w\rangle \quad \forall v, w \in \mathbb{C}^n$$

Then π uniquely extends to a unital morphism of algebras,

$$c : \mathbb{C}l_{\mathbb{C}}(\mathbb{R}^n) \to A.$$

In addition, for any $I = (i_1, \ldots, i_l) \in \mathcal{I}_n$,

$$\pi(c(e^{i_1} \wedge \cdots \wedge e^{i_l})) = \pi(c(e^{i_1})) \cdots \pi(c(e^{i_l})).$$

<u>Proof</u>: The unicity follows from the fact that the $c(v)$'s, $v \in \mathbb{C}^n$, generate $\mathbb{C}l_{\mathbb{C}}(\mathbb{R}^n)$. Therefore, we only have to show that the map $\pi : c(\mathbb{C}^n) \to A$ can be extended into a morphism of algebras $\pi : \mathbb{C}l_{\mathbb{C}}(\mathbb{R}^n) \to A$.

We extend π into the $\mathbb{C}$-linear map $\pi : \mathbb{C}l_{\mathbb{C}}(\mathbb{R}^n) \to A$ s.t., for all $I = (i_1, \ldots, i_l) \in \mathcal{I}_n$,

$$\pi(1) = 1_A$$

$$\pi(c(e^{i_1} \wedge \cdots \wedge e^{i_l})) = \pi(c(e^{i_1})) \cdots \pi(c(e^{i_l})) \quad \forall I = (i_1, \ldots, i_l) \in \mathcal{I}_n \setminus \emptyset.$$

As $c(e^{i_1} \wedge \cdots \wedge e^{i_l}) = c(e^{i_1}) \cdots c(e^{i_l})$, we see that, by definition,

$$\pi(c(e^{i_1}) \cdots c(e^{i_l})) = \pi(c(e^{i_1})) \cdots \pi(c(e^{i_l})) \quad \text{if } 1 \leq i_1 < \cdots < i_l \leq n.$$

Observe also that (4.1) implies that

$$\pi(c(e^j))\pi(c(e^l)) + \pi(c(e^l))\pi(c(e^j)) = -2\delta^{jl},$$

and hence

$$(4.2) \quad \pi(c(e^j))\pi(c(e^l)) = -\pi(c(e^l))\pi(c(e^j)) \quad \text{if } j \neq l$$

$$(4.3) \quad \pi(c(e^j))^2 = -1$$

<u>Claim</u>: Let $v \in \mathbb{C}^n$ and $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$. Then
$$\pi(c(v)a) = \pi(c(v))\pi(a)$$
<u>Proof of the claim</u>: It is enough to prove the claim when $v = e^j$ and $a = c(e^I)$ for some $j \in \{1, \dots, n\}$ and $I = (i_1, \dots, i_\ell) \in \mathcal{I}_n$.

Assume that $j \notin I$. If $j < i_1$, then
$$\begin{aligned}\pi(c(e^j)c(e^I)) &= \pi(c(e^j)c(e^{i_1})\cdots c(e^{i_\ell}))\\ &= \pi(c(e^j))\pi(c(e^{i_1}))\cdots\pi(c(e^{i_\ell}))\\ &= \pi(c(e^j))\pi(c(e^I)).\end{aligned}$$
Suppose that $j > i_1$ and let $p = \sup\{q;\ i_q < j\}$. Then
$$\begin{aligned}c(e^j)c(e^I) &= c(e^j)c(e^{i_1})\cdots c(e^{i_\ell})\\ &= (-1)^p c(e^{i_1})\cdots c(e^{i_p})c(e^j)\cdots c(e^{i_\ell})\end{aligned}$$
Thus,
$$\pi(c(e^j)c(e^I)) = (-1)^p \pi(c(e^{i_1}))\cdots\pi(c(e^{i_p}))\pi(c(e^j))\cdots\pi(c(e^{i_\ell})).$$
Therefore, using (4.2) we get
$$\pi(c(e^j)c(e^I)) = \pi(c(e^j))\pi(c(e^{i_1}))\cdots\pi(c(e^{i_\ell})) = \pi(c(e^j))\pi(c(e^I)).$$
Assume now that $j \in I$ and let p the index s.t. $i_p = j$. Then
$$\begin{aligned}c(e^j)c(e^I) &= c(e^{i_p})c(e^{i_1})\cdots c(e^{i_\ell})\\ &= (-1)^{p-1}c(e^{i_1})\cdots c(e^{i_p})c(e^{i_p})\cdots c(e^{i_\ell})\\ &= (-1)^p c(e^{i_1})\cdots \widehat{c(e^{i_p})}\cdots c(e^{i_\ell}).\end{aligned}$$
Likewise, by using (4.2)-(4.3) we get
$$\begin{aligned}\pi(c(e^j))\pi(c(e^I)) &= \pi(c(e^{i_p}))\pi(c(e^{i_1}))\cdots\pi(c(e^{i_\ell}))\\ &= (-1)^p\pi(c(e^{i_1}))\cdots\widehat{\pi(c(e^{i_p}))}\cdots\pi(c(e^{i_\ell})).\end{aligned}$$
As $\pi(c(e^{i_1})\cdots\widehat{c(e^{i_p})}\cdots c(e^{i_\ell})) = \pi(c(e^{i_1}))\cdots\pi(\widehat{c(e^{i_p})})\cdots\pi(c(e^{i_\ell}))$, we deduce that
$$\pi(c(e^j)c(e^I)) = \pi(c(e^j))\pi(c(e^I)).$$
This completes the proof of the claim. □

Let us go back to the proof of Lemma 4.2. Let $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$. Then combining the above claim with an easy induction shows that
$$\pi(c(e^{i_1})\cdots c(e^{i_\ell})a) = \pi(c(e^{i_1}))\cdots\pi(c(e^{i_\ell}))\pi(a) \quad \text{if } i_1 < \cdots < i_\ell.$$

As $c(e^{i_1})\cdots c(e^{i_k}) = c(e^{i_1}\wedge\cdots\wedge e^{i_k})$ and $\pi(c(e^{i_1}))\cdots\pi(c(e^{i_k})) = \pi(c(e^{i_1}\wedge\cdots\wedge e^{i_k}))$, we see that, for all $I \in \mathcal{I}_n$,
$$\pi(c(e^I)a) = \pi(c(e^I))\,\pi(a).$$
By linearity it follows that
$$\pi(ba) = \pi(b)\,\pi(a) \qquad \forall a, b \in Cl_0(\mathbb{R}^n)$$
This proves that π is a morphism of algebras. □

Let us go back to the construction of the spinor representation. Using the above we see that the linear map $\rho: c(\Lambda^n) \longrightarrow \mathrm{End}(\$)$ uniquely extends to a morphism of algebras
$$\rho: Cl_0(\mathbb{C}^n) \longrightarrow \mathrm{End}(\$),$$
in such way that
$$\rho(c(e^{i_1}\wedge\cdots\wedge e^{i_k})) = \rho(c(e^{i_1}))\cdots\rho(c(e^{i_k})) \text{ if } i_1<\cdots<i_k.$$
In particular, ρ is a representation of $Cl_0(\mathbb{R}^n)$ in $\$$.

B.
Irreducibility of the spinor representation (n even):

Let $\mathcal{I}$ be the family consisting of the empty subset of $\{1,\dots,m\}$ and of all the ordered subsets $I=(i_1,\dots,i_m)$ of $\{1,\dots,m\}$ w/ $1\leq i_1<\dots<i_m\leq l$. For $I\in\mathcal{I}$ we set

$$z^{I}=z^{i_1}\wedge\cdots\wedge z^{i_m} \quad \text{and} \quad z^{\bar I}=z^{\bar i_1}\wedge\cdots\wedge z^{\bar i_q} \quad \text{if } I=(i_1,\dots,i_q)\neq\emptyset,$$

$$z^{\emptyset}=z^{\bar\emptyset}=1.$$

Moreover, the length of any $I\in\mathcal{I}$ is defined to be

$$|I|=\begin{cases}0 & \text{if } I=\emptyset,\\ q & \text{if } I=(i_1,\dots,i_q)\neq\emptyset.\end{cases}$$

Then:

- $\{z^{I}\wedge z^{\bar J}\}_{I,J\in\mathcal{I}}$ is an orthonormal basis of $\dot\Lambda\mathbb{C}^n$,
- $\{z^{\bar I}\}_{I\in\mathcal{I}}$ is an orthonormal basis of $\$=\Lambda^{0,*}\mathbb{C}^n$,
- $\{z^{\bar I}\}_{I\in\mathcal{I},\, |I|=q}$ is an orthonormal basis of $\Lambda^{0,q}\mathbb{C}^n$.

Furthermore, if $I=(i_1,\dots,i_q)\neq\emptyset$, then

$$c(z^{I})=c(z^{i_1})\cdots c(z^{i_q}) \quad \text{and} \quad c(z^{\bar I})=c(z^{\bar i_1})\cdots c(z^{\bar i_q}).$$

Claim. Let $\bar z, \bar w\in\$\setminus\{0\}$. Then there always exists $a\in Cl_{\mathbb{C}}(\mathbb{R}^n)$ s.t.

$$\rho(a)\bar z=\bar w.$$

Proof of the claim: Write $\bar z=\sum\alpha_I z^{\bar I}$, $\alpha_I\in\mathbb{C}$, and define

$$q=\sup\{|I|;\ \alpha_I\neq 0\}.$$

Assume that $q=0$. Then $\bar z=\alpha.1$, with $\alpha=\alpha_\emptyset\in\mathbb{C}\setminus\{0\}$. Set $a_z=\alpha^{-1}$. Then

$$\rho(a)\bar z=\rho(\alpha^{-1})\alpha=\alpha^{-1}\cdot\alpha=1.$$

Suppose that $q\geq 1$ and let $I^0=(i_1,\dots,i_q)\in\mathcal{I}$ be s.t. $|I^0|=q$ and $\alpha_{I^0}\neq 0$. Observe that

$$i(z^{i_q})\cdots i(z^{i_1})\, z^{\bar I}=\begin{cases}0 & \text{if } |I|<q \text{ or } |I|=q \text{ and } I\neq I^0,\\ 1 & \text{if } I=I^0.\end{cases}$$

Set

$$a_z=\alpha_{I_0}^{-1}\left(\frac{-1}{\sqrt2}\right)^q c(z^{i_q})\cdots c(z^{i_1})=\alpha_{I_0}^{-1}\left(\frac{-1}{\sqrt2}\right)^q c(z^{i_q}\wedge\cdots\wedge z^{i_1})$$

Then

$$\rho(a_z)\bar z=\left\{\alpha_{I_0}^{-1}\left(\frac{-1}{\sqrt2}\right)^q\left(\frac{1}{\sqrt2}\right)i(z^{i_q})\cdots\left(\frac{-1}{\sqrt2}\right)c(z^{i_1})\right\}\sum_{|I|\leq q}\alpha_I z^{\bar I}$$

$$=\alpha_{I_0}^{-1}\sum_{|I|\leq q}\alpha_I\, c(z^{i_q})\cdots c(z^{i_1})\, z^{\bar I}=\alpha_{I_0}^{-1}\alpha_{I_0}=1.$$

Therefore, whether $q=0$ or $q\geq 1$ there always exists $a_2 \in Cl_{\mathbb{C}}(\mathbb{R}^n)$ s.t.

$$\rho(a_2)\bar{z} = 1.$$

Next, let us write $\bar{\omega} = \sum \beta_I z^I$ and define

$$a_1 = \sum 2^{-|I|/2} \beta_I \, c(z^{\bar{I}}).$$

Observe that if $I = (i_1, \ldots, i_q) \in \mathcal{J} \setminus \{\phi\}$, then

$$\begin{aligned}
\rho(c(z^{\bar{I}})) &= \rho\left(c(z^{\bar{i}_1}) \cdots c(z^{\bar{i}_q})\right) \\
&= \rho\left(c(z^{\bar{i}_1})\right) \cdots \rho\left(c(z^{\bar{i}_q})\right) \\
&= \sqrt{2}\,\varepsilon(z^{\bar{i}_1}) \cdots \sqrt{2}\,\varepsilon(z^{\bar{i}_q}) \\
&= 2^{q/2}\,\varepsilon(z^{\bar{i}_1} \wedge \cdots \wedge z^{\bar{i}_q}) \\
&= 2^{|I|/2}\,\varepsilon(z^{\bar{I}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\rho(a_1) &= \sum 2^{-|I|/2} \cdot \beta_I \cdot \rho\left(c(z^{\bar{I}})\right) \\
&= \sum 2^{-|I|/2} \cdot \beta_I \cdot 2^{|I|/2} \varepsilon(z^{\bar{I}}) \\
&= \varepsilon\left(\sum \beta_I z^{\bar{I}}\right) \\
&= \varepsilon(\bar{\omega}).
\end{aligned}$$

Now, define

$$a = a_1 a_2.$$

As $\rho(a_2)\bar{z} = 1$ and $\rho(a_1) = \varepsilon(\bar{\omega})$, we see that

$$\rho(a)\bar{z} = \rho(a_1)\rho(a_2)\bar{z} = \varepsilon(\bar{\omega}).1 = \bar{\omega}.$$

The claim is thus proved. $\square$

Now, let E be a subspace of $\$$ which is invariant under ρ and suppose that $E \neq \{0\}$. Let $\bar{\omega} \in \$ \setminus \{0\}$ and let $\bar{z} \in E \setminus \{0\}$. Then by Claim 1 there exists $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$ s.t. $\rho(a)\bar{z} = \bar{\omega}$. As E is ρ-invariant we see that $\bar{\omega}$ must be contained. It then follows that $E = \$$. Thus ρ has no non-trivial invariant subspaces, that is, ρ is an <u>irreducible representation</u>.

C. Invertibility of the spinor representation (more):

As $\{z^{\bar{I}}\wedge z^{J}\}_{I,J\in\mathcal{J}}$ is a basis of $\Lambda^{*}\mathbb{C}^{n}$, we see that $\{c(z^{\bar{I}}\wedge z^{J})\}_{I,J\in\mathcal{J}}$ is a basis of $Cl_{\mathbb{C}}(\mathbb{R}^{n})$. Observe also that as, for all $\xi\in\Lambda^{i}\mathbb{C}^{n}$ and $\eta\in\Lambda^{j}\mathbb{C}^{n}$ at $i,j\leq n$

$$c(\xi\wedge\eta)=c(\xi)c(\eta) \mod c(\Lambda^{i+j-2}\mathbb{C}^{n}),$$

we see that, for all $I,J\in\mathcal{J}$,

$$c(z^{\bar{I}}\wedge z^{J})=c(z^{\bar{I}})c(z^{J}) \mod c(\Lambda^{|I|+|J|-2}\mathbb{C}^{n}).$$

Therefore, upon using a suitable order of the elements of $\mathcal{J}$, the matrix of the coordinates of the $c(z^{\bar{I}})c(z^{J})$ with respect to the basis $\{c(z^{\bar{I}}\wedge z^{J})\}_{I,J\in\mathcal{J}}$ can be written as an upper-triangular matrix whose diagonal entries are all equal to 1. In particular, this matrix is invertible and hence $\{c(z^{\bar{I}})c(z^{J})\}_{I,J\in\mathcal{J}}$ is a basis of $Cl_{\mathbb{C}}(\mathbb{R}^{n})$. Therefore, any $a\in Cl_{\mathbb{C}}(\mathbb{R}^{n})$ can be uniquely written as

$$a=\sum\alpha_{I,J}\,c(z^{\bar{I}})c(z^{J}),\qquad \alpha_{I,J}\in\mathbb{C}.$$

Claim 2: Let $a\in Cl_{\mathbb{C}}(\mathbb{R}^{n})\setminus\{0\}$. Then there always exists $\sigma\in\mathcal{S}$ s.t.

$$\rho(c(a))\sigma\neq 0.$$

Proof: let us write

$$a=\sum\alpha_{I,J}\,c(z^{\bar{I}})c(z^{J}),\qquad \alpha_{I,J}\in\mathbb{C},$$

and define

$$p_0=\inf\{|J|;\ \exists I\in\mathcal{J}\text{ s.t. }\alpha_{I,J}\neq 0\}$$

Observe also that if $J=(j_1,\ldots,j_p)\in\mathcal{J}\setminus\{\phi\}$, then

$$c(z^{J})=c(z^{j_1})\cdots c(z^{j_p})$$

$$\rho(c(z^{J}))=\rho(c(z^{j_1}))\cdots\rho(c(z^{j_p}))$$
$$=(-\sqrt{2})^{p}\,\iota(z^{j_1})\cdots\iota(z^{j_p})$$

Suppose now that $p_0=0$. Then, for all $J=(j_1,\ldots,j_p)\in\mathcal{J}\setminus\{\phi\}$,

$$\rho(c(z^{J})).1=(-\sqrt{2})^{p}\,\iota(z^{j_1})\cdots\iota(z^{j_p})1=0$$

Thus,

$$\rho(a)1=\sum\alpha_{I,J}\,\rho(c(z^{\bar{I}}))\rho(c(z^{J}))1$$
$$=\sum_{I\in\mathcal{J}}\alpha_{I,\phi}\,\rho(c(z^{\bar{I}}))\rho(c(1))1$$
$$=\sum_{I\in\mathcal{J}}\alpha_{I,\phi}\,\rho(c(z^{\bar{I}}))1$$

As observed before,

$$\rho(c(z^{\overline{I}})) = 2^{-|I|/2} \varepsilon(z^{\overline{I}}).$$

Therefore,

$$\rho(a)1 = \sum_{I \in \mathcal{J}} \alpha_{I,\phi} 2^{-|I|/2} \varepsilon(z^{\overline{I}})1 = \sum_{I \in \mathcal{J}} \alpha_{I,\phi} z^{\overline{I}} \neq 0,$$

since at least one of the $\alpha_{I,\phi}$ is $\neq 0$ due to the fact that $p_0 = 0$. This proves the claim when $p_0 = 0$.

Suppose now that p_0 is ≥ 1 and let $J_0 = (j_1, \dots, j_{p_0}) \in \mathcal{J} \setminus \{\phi\}$ be such that $\exists J \in \mathcal{J}$ so that $\alpha_{I,J_0} \neq 0$. Let $J = (j_1, \dots, j_p) \in \mathcal{J} \setminus \{\phi\}$ with $p \geq p_0$. Then

$$\begin{aligned}
\rho(c(z^J))\, z^{\overline{j_{p_0}}} \wedge \dots \wedge z^{\overline{j_1}} &= (-\sqrt{2})^{p} i(z^{j_1}) \cdots i(z^{j_p})\, z^{\overline{j_{p_0}}} \wedge \dots \wedge z^{\overline{j_1}} \\
&= \begin{cases} 0 & \text{if } p > p_0 \text{ or if } p = p_0 \text{ and } J \neq J_0, \\ (-\sqrt{2})^{p_0} & \text{if } J = J_0. \end{cases}
\end{aligned}$$

Therefore, if we set

$$\sigma = \left(-\frac{1}{\sqrt{2}}\right)^{p_0} z^{\overline{j_{p_0}}} \wedge \dots \wedge z^{\overline{j_1}},$$

then we have

$$\begin{aligned}
\rho(a)\sigma &= \sum_{\substack{I, J \in \mathcal{J} \\ |J| \geq p_0}} \alpha_{I,J}\, \rho(c(z^{\overline{I}}))\, \rho(c(z^J)) \cdot \left(-\frac{1}{\sqrt{2}}\right)^{p_0} z^{\overline{j_{p_0}}} \wedge \dots \wedge z^{\overline{j_1}} \\
&= \sum_{I \in \mathcal{J}} \alpha_{I,J_0}\, \rho(c(z^{\overline{I}})) \cdot 1 \\
&= \sum_{I \in \mathcal{J}} \alpha_{I,J_0} z^{\overline{I}} \quad (\text{as in the case } p_0 = 0) \\
&\neq 0.
\end{aligned}$$

Thus the claim holds even when $p_0 \neq 0$. The proof of the claim is complete.

This claim shows that, for all $a \in Cl_{\mathbb{C}}(\mathbb{R}^n) \setminus \{0\}$, the endomorphism $\rho(a)$ is $\neq 0$. Thus,

$$\ker \rho = \{0\}.$$

Observe now that $\$ = \bigoplus_{q=0}^{m} \Lambda^{0,q} \mathbb{C}^n$ has dimension $2^m = 2^{n/2}$, and so $\dim \mathrm{End}(\$_n) = (2^{n/2})^2 = 2^n$. Moreover, the isomorphism between $Cl_{\mathbb{C}}(\mathbb{R}^n)$ and $\Lambda^{*}\mathbb{C}^n$ shows that $\dim Cl_{\mathbb{C}}(\mathbb{R}^n) = 2^n$. Since $Cl_{\mathbb{C}}(\mathbb{R}^n)$ and $\mathrm{End}(\$)$ have same dimension, the injectivity of ρ implies that ρ is an isomorphism.

D.

Uniqueness of the spinor representation (n even):

Before getting to the uniqueness of the spinor representation, let us briefly recall why the matrix algebras are simple, i.e., they have only one irreducible representation up to equivalence.

Let $k \in \mathbb{N}$ and let us denote by $\lambda : M_k(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^k)$ the representation of $M_k(\mathbb{C})$ in $\mathbb{C}^k$ defined by

$$\lambda(A)\xi := A.\xi \qquad \forall A \in M_k(\mathbb{C}) \quad \forall \xi \in \mathbb{C}^k.$$

This representation is irreducible because, for all $\xi, \eta \in \mathbb{C}^k \setminus \{0\}$, there is $A \in M_k(\mathbb{C})$ s.t. $A\xi = \eta$.

Prop. 43: Up to equivalence, λ is the unique irreducible representation of $M_k(\mathbb{C})$.

Proof: Let $\pi : M_k(\mathbb{C}) \to \mathrm{End}(V)$ be an irreducible representation of $M_k(\mathbb{C})$ in some vector space V.
For $\xi, \eta \in \mathbb{C}^k$ we shall denote by $E_{\xi,\eta}$ the elementary matrix s.t.

$$(E_{\xi,\eta})\zeta = \langle \eta, \zeta \rangle \xi \qquad \forall \zeta \in \mathbb{C}^k.$$

Then $M_k(\mathbb{C})$ is spanned by the $E_{\xi,\eta}$ as ξ and η range over $\mathbb{C}^k \setminus \{0\}$ (or over a basis of $\mathbb{C}^k$).
Furthermore, observe that

$$A E_{\xi,\eta} = E_{A\xi,\eta} \qquad \forall A \in M_k(\mathbb{C}) \quad \forall \xi, \eta \in \mathbb{C}^k.$$

As π is non-zero, there exist $\xi_0, \eta_0 \in \mathbb{C}^k \setminus \{0\}$ and $v \in V \setminus \{0\}$ s.t.

$$\pi(E_{\xi_0,\eta_0})v \neq 0.$$

Let $\varphi : \mathbb{C}^k \to V$ be the linear map defined by

$$\varphi(\xi) := \pi(E_{\xi,\eta_0})v \qquad \forall \xi \in \mathbb{C}^k$$

Let $\zeta \in \mathbb{C}^k$ and $A \in M_k(\mathbb{C})$, then

$$\varphi(\lambda(A)\xi) = \varphi(A\xi) = \pi(E_{A\xi,\eta_0})v$$
$$= \pi(AE_{\xi,\eta_0})v = \pi(A)\pi(E_{\xi,\eta_0})v = \pi(A)\varphi(\xi).$$

Thus φ is an $M_k(\mathbb{C})$-equivariant linear map. This implies that $\ker\varphi$ and $\operatorname{im}\varphi$ are invariant subspaces of λ and π respectively. As λ and π are irreducible representations and φ is non-zero (since $\varphi(\xi_0) = \pi(E_{\xi_0,\eta_0})v \neq 0$), we then deduce that $\ker\varphi = \{0\}$ and $\operatorname{im}\varphi = V$, that is, φ is an $M_k(\mathbb{C})$-equivariant linear isomorphism. Therefore, the representations λ and π are equivalent, proving the proposition. $\square$

The above proposition means that $M_k(\mathbb{C})$ is a simple algebra. By extension any algebra isomorphic (as algebra) to some $M_k(\mathbb{C})$ is simple.

Now, as we have isomorphisms of algebras,

$$Cl_{\mathbb{C}}(\mathbb{R}^n) \simeq \mathrm{End}(\$) \simeq M_k(\mathbb{R}^n) \quad \text{w/ } k = 2^{n/2},$$

we see that the Clifford algebra $Cl_{\mathbb{C}}(\mathbb{R}^n)$ is simple. Therefore, the spinor representation is the unique irreducible representation of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ up to equivalence. This completes the proof of the Theorem. $\square$.

When n odd, the following holds.

Theorem 4.4 (Spinor Representation, odd case): Assume n odd. Then

(1) Up to equivalence, there is a unique irreducible representation of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ mapping Γ to the identity, namely, the spinor representation,

$$\rho: Cl_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow \$_n,$$

where $\$_n$ is the spinor bundle of $\mathbb{R}^n$ (see definition below).

(2) The spinor representation induces an isomorphism of algebras,

$$Cl_{\mathbb{C}}^{+}(\mathbb{R}^n) \simeq \mathrm{End}(\$_n).$$

(3) Under the spinor representation, the chirality operator Γ acts like the identity on $\$_n$.

Remark: As we shall see, $\$_n$ has dimension $2^{\frac{n-1}{2}}$, so the isomorphism $Cl_{\mathbb{C}}^{+}(\mathbb{R}^n) \simeq \mathrm{End}(\$_n)$ and the equality $Cl_{\mathbb{C}}(\mathbb{R}^n) = Cl_{\mathbb{C}}^{+}(\mathbb{R}^n) \oplus \Gamma Cl_{\mathbb{C}}^{+}(\mathbb{R}^n)$ imply that

$$Cl_{\mathbb{C}}^{+}(\mathbb{R}^n) \simeq M_{2^{\frac{n-1}{2}}}(\mathbb{R}^n) \oplus \Gamma M_{2^{\frac{n-1}{2}}}(\mathbb{R}^n).$$

Proof of the Theorem 4.4: Let us denote by $\gamma: Cl_{\mathbb{C}}(\mathbb{R}^{n-1}) \longrightarrow Cl_{\mathbb{C}}^{+}(\mathbb{R}^n)$ the algebra isomorphism from $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$ onto $Cl_{\mathbb{C}}^{+}(\mathbb{R}^n)$ as constructed on pages 2 and 3. Let $\$_{n-1}$ be the space of spinors of $\mathbb{R}^n$ and let us denote by $\rho_{n-1}: Cl_{\mathbb{C}}(\mathbb{R}^{n-1}) \longrightarrow \mathrm{End}(\$_{n-1})$ the spinor representation of $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$.

Set $\$_{n-1} = \$_n$ and let $\rho_n: Cl_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow \mathrm{End}(\$_n)$ be the linear map defined by

$$\rho_n(a' + \Gamma a'') = \rho_{n-1}(\gamma^{-1}(a')) + \rho_{n-1}(\gamma^{-1}(a'')) \quad \forall a', a'' \in Cl_{\mathbb{C}}^{+}(\mathbb{R}^n).$$

Observe that $\rho_n(\Gamma) = \mathrm{id}_{\$_n}$.

Let $a, b \in Cl_{\mathbb{C}}(\mathbb{R}^n)$ and write $a = a' + \Gamma a''$ and $b = b' + \Gamma b''$ with a', a'', b', b'' in $Cl_{\mathbb{C}}^{+}(\mathbb{R}^n)$. Then, as $\Gamma^2 = 1$ and Γ commutes with all the elements of $Cl_{\mathbb{C}}(\mathbb{R}^n)$,

$$ab = (a' + \Gamma a'')(b' + \Gamma b'') = (a'b' + a''b'') + \Gamma(a'b'' + a''b').$$

Thus,

$$\rho_n(ab) = \rho_n\big((a'b'+a''b'') + \Gamma(a'b''+a''b')\big) = \rho_{n-1}\circ\gamma^{-1}(a'b'+a''b'') + \rho_{n-1}\circ\gamma^{-1}(a'b''+a''b')$$
$$= \rho_{n-1}\circ\gamma^{-1}(a')\rho_{n-1}\circ\gamma^{-1}(b') + \rho_{n-1}\circ\gamma^{-1}(a'')\rho_{n-1}\circ\gamma^{-1}(b'') + \rho_{n-1}\circ\gamma^{-1}(a')\rho_{n-1}\circ\gamma^{-1}(b'') + \rho_{n-1}\circ\gamma^{-1}(a'')\rho_{n-1}\circ\gamma^{-1}(b')$$
$$= \rho_n(a')\rho_{n}(b') + \rho_n(a'')\rho_n(b'') + \rho_n(a')\rho_n(b'') + \rho_n(a'')\rho_n(b')$$
$$= \big(\rho_n(a') + \rho_n(a'')\big)\big(\rho_n(b') + \rho_n(b'')\big)$$
$$= \rho_n(a'+\Gamma a'')\rho_n(b'+\Gamma b'') = \rho_n(a)\rho_n(b).$$

Therefore ρ_n is a representation of the algebra $Cl_{\mathbb{C}}(\mathbb{R}^n)$ in $\$_n$.

Observe that on the subalgebra $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$, the spinor representation ρ_n agrees with $\rho_{n-1}\circ\gamma^{-1}$, which is the spinor representation of $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$ under the algebra isomorphism $\gamma: Cl_{\mathbb{C}}(\mathbb{R}^{n-1}) \simeq Cl_{\mathbb{C}}^+(\mathbb{R}^n)$. As the representation ρ_{n-1} is irreducible, it then follows that $\$_n$ has no non-trivial subspaces invariant under $\rho_n\big(Cl_{\mathbb{C}}^+(\mathbb{R}^n)\big)$, and hence ρ_n has no non-trivial invariant subspaces. That is, ρ_n is an irreducible representation.

Let us now prove the uniqueness of the spinor representation ρ_n. Let $\pi: Cl_{\mathbb{C}}(\mathbb{R}^n) \to End(V)$ be an irreducible representation of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ in some vector space V in such way that $\pi(\Gamma) = id_V$.

Let W be a subspace of V which is invariant under the action of $\pi\big(Cl_{\mathbb{C}}^+(\mathbb{R}^n)\big)$, i.e.,

$$\pi(a)w \in W \quad \forall a \in Cl_{\mathbb{C}}^+(\mathbb{R}^n) \quad \forall w \in W.$$

Let $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$ and let us write $a = a' + \Gamma a''$ w/ $a', a'' \in Cl_{\mathbb{C}}^+(\mathbb{R}^n)$. As $\pi(\Gamma) = id_V$, we see that

$$\pi(a) = \pi(a'+\Gamma a'') = \pi(a') + \pi(\Gamma)\pi(a'') = \pi(a') + \pi(a'') = \pi(a'+a'').$$

As $a'+a'' \in Cl_{\mathbb{C}}^+(\mathbb{R}^n)$, we deduce that, for all $w \in W$,

$$\pi(a)w = \pi(a'+a'')w \in W.$$

Therefore, W is an invariant subspace of π, and hence W is either $\{0\}$ or V. Thus, we have shown that $\pi|_{Cl_{\mathbb{C}}^+(\mathbb{R}^n)}$ is an irreducible representation of $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$.

Observe that, as ρ_{n-1} is an algebra isomorphism from $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$ onto $End(\$_{n-1}) = End(\$_n)$, and $\rho_n|_{Cl_{\mathbb{C}}^+(\mathbb{R}^n)} = \rho_{n-1}\circ\gamma^{-1}$, we see that ρ_n induces an algebra isomorphism,

$$Cl_{\mathbb{C}}^+(\mathbb{R}^n) \simeq End(\$_n)$$

As $\dim \$_n = \dim \$_{n-1} = 2^{\frac{n-1}{2}}$, we thus have an algebra isomorphism,

$$Cl_{\mathbb{C}}^+(\mathbb{R}^n) \simeq M_{2^{\frac{n-1}{2}}}(\mathbb{C}).$$

It then follows that, up to equivalence, $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$ has a unique irreducible representation. Thus

$\rho_n|Cl_{\mathbb{C}}^+(\mathbb{R}^n)$ and $\pi|Cl_{\mathbb{C}}^+(\mathbb{R}^n)$ are equivalent representations of $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$, i.e., $\exists$ linear isomorphism $\varphi: \$_n \longrightarrow V$ s.t. $\varphi(\rho_n(a)\sigma) = \pi(a)\varphi(\sigma)$ $\forall \sigma \in \$_n$ $\forall a \in Cl_{\mathbb{C}}^+(\mathbb{R}^n)$.

Let $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$ and write $a = a' + \Gamma a''$. We have

$$\rho_n(a) = \rho_n(a' + \Gamma a'') = \rho_n(a' + a'') \quad \text{and} \quad \pi(a) = \pi(a' + \Gamma a'') = \pi(a' + a'').$$

As $a' + a'' \in Cl_{\mathbb{C}}^+(\mathbb{R}^n)$, we deduce that, for all $\sigma \in \$_n$,

$$\varphi(\rho_n(a)\sigma) = \varphi(\rho_n(a' + a'')\sigma) = \pi(a' + a'')\varphi(\sigma) = \pi(a)\varphi(\sigma).$$

This proves that φ is an $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant linear isomorphism from $\$_n$ onto V, so π and ρ_n are equivalent representations of $Cl_{\mathbb{C}}(\mathbb{R}^n)$. The proof of the Theorem is complete. $\square$

As $Cl(\mathbb{R}^n)$ is a subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet \mathbb{R}^n)$, we can regard $Cl_{\mathbb{C}}(\mathbb{R}^n)$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda^\bullet \mathbb{C}^n)$. It then follows from Lemma 3.1 that $Cl_{\mathbb{C}}(\mathbb{R}^n)$ is an involutive subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda^\bullet \mathbb{C}^n)$.

<u>Prop. 4.5:</u> The spinor representation is a $*$-representation, i.e.,

$$\rho(a^*) = \rho(a)^* \qquad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

<u>Proof:</u> Let us first assume that n is even.

Using Lemma 2.3 we see that, for $j = 1, \dots, m$,

$$\varepsilon(z^{\bar{j}})^* = (\varepsilon(e^j) - i\,\varepsilon(e^{j+m}))^* = \varepsilon(e^j)^* + i\,\varepsilon(e^{m+j})^*$$
$$= \iota(e^j) + i.\,\iota(e^{m+j}) = \iota(z^j).$$

Thus,

$$\rho(c(z^{\bar{j}}))^* = \sqrt{2}\,\varepsilon(z^{\bar{j}})^* = \sqrt{2}\,\iota(z^{\bar{j}}) = -\rho(c(z^j))$$
$$\rho(c(z^j))^* = -\rho(c(z^{\bar{j}})).$$

Moreover, using Lemma 3.1 we also get

$$c(z^{\bar{j}})^* = (c(e^j) - i\,c(e^{m+j}))^* = c(e^j)^* + i\,c(e^{m+j})^*$$
$$= -c(e^j) - i\,c(e^{m+j}) = -c(z^j),$$

and hence $c(z^j)^* = -c(z^{\bar{j}})$. Thus,

$$\rho(c(z^{\bar{j}}))^* = \rho(-c(z^j)) = \rho(c(z^{\bar{j}})^*)$$
$$\rho(c(z^j)) = \rho(-c(z^{\bar{j}})) = \rho(c(z^j)).$$

It then follows that

$$\rho(c(v)^*) = \rho(c(v)^*) \qquad \forall v \in \mathbb{C}^n.$$

Therefore, for any $I=(i_1,\ldots,i_\ell)\in\mathcal{I}_n$,

$$\begin{aligned}\rho(c(e^I)^*) = \rho\big((c(e^{i_1})\cdots c(e^{i_\ell}))^*\big) &= \rho(c(e^{i_\ell})^*\cdots c(e^{i_1})^*)\\ &= \rho(c(e^{i_\ell})^*)\cdots\rho(c(e^{i_1})^*)\\ &= \rho(c(e^{i_\ell}))^*\cdots\rho(c(e^{i_1}))^*\\ &= \big(\rho(c(e^{i_1}))\cdots\rho(c(e^{i_\ell}))\big)^*\\ &= \rho(c(e^I))^*.\end{aligned}$$

It then follows that $\rho(a^*)=c(a)^*$ $\forall a\in Cl_{\mathbb{C}}(\mathbb{R}^n)$. This proves the proposition when n is even.

Let us now assume that n is odd. Let $\varphi: Cl_{\mathbb{C}}(\mathbb{R}^{n-1})\longrightarrow Cl^+_{\mathbb{C}}(\mathbb{R}^n)$ be the linear isomorphism

$$\varphi(a) = a^+ + a^- c(e^n),$$

where $e^n=(0,\ldots,0,1)$ and $a=a^++a^-$ is the decomposition in terms of components in $Cl^{\pm}_{\mathbb{C}}(\mathbb{R}^n)$. Then by definition,

$$\rho(a) = \rho_{n-1}(\varphi^{-1}(a)) \quad \forall a\in Cl^+(\mathbb{R}^n),$$

where ρ_{n-1} is the spinor representation of $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$.

Observe that Lemma 3.1 implies that $Cl^+_{\mathbb{C}}(\mathbb{R}^{n-1})$ and $Cl^-_{\mathbb{C}}(\mathbb{R}^{n-1})$ are involutive subspaces of $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$. Thus,

$$(a^\pm)^* = (a^*)^\pm \qquad \forall a\in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

As pointed out in the proof of Lemma 3.8, all the elements of $Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$ anticommutes with $c(e^n)$. As $c(e^n)^*=-c(e^n)$ we see that, for all $a\in Cl_{\mathbb{C}}(\mathbb{R}^{n-1})$,

$$\begin{aligned}\varphi(a)^* = (a^+ + a^- c(e^n))^* &= (a^+)^* + c(e^n)^*(a^-)^*\\ &= (a^*)^+ - c(e^n)(a^*)^- = (a^*)^+ + (a^*)^- c(e^n) = \varphi(a^*).\end{aligned}$$

This proves that φ is a $*$-isomorphism, so φ^{-1} is a $*$-isomorphism. Therefore, for all $a\in Cl^+_{\mathbb{C}}(\mathbb{R}^n)$,

$$\rho(a^*) = \rho_{n-1}(\varphi^{-1}(a^*)) = \rho_{n-1}(\varphi^{-1}(a)^*) = \rho_{n-1}(\varphi^{-1}(a))^* = \rho(a)^*.$$

Let $a\in Cl^-_{\mathbb{C}}(\mathbb{R}^n)$. Then $\Gamma a\in Cl^+_{\mathbb{C}}(\mathbb{R}^n)$. As $\rho(\Gamma)=1$ we get

$$\rho((\Gamma a)^*) = \rho(\Gamma a)^* = (\rho(\Gamma)\rho(a))^* = \rho(a)^*.$$

As Lemma 3.10 tells us that $\Gamma^*=\Gamma$ we also get

$$\rho((\Gamma a)^*) = \rho(a^*\Gamma) = \rho(a^*)\rho(\Gamma) = \rho(a^*),$$

and hence $\rho(a^*)=\rho(a)^*$.

All this shows that ρ is a $*$-representation even when n is odd. The proof is complete. $\square$

From now on and until the end of the section, we assume that n is <u>even</u>.

As $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$, we see that

$$\rho(\Gamma)^2 = \rho(\Gamma^2) = 1 \quad \text{and} \quad \rho(\Gamma)^* = \rho(\Gamma^*) = \rho(\Gamma).$$

Therefore, we get have an orthogonal decomposition

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-,$$

given by the eigenspaces of $\rho(\Gamma)$, i.e.,

$$\mathcal{S}^+ = \ker(\rho(\Gamma) - 1) \quad \text{and} \quad \mathcal{S}^- = \ker(\rho(\Gamma)+1).$$

We thus have a $\mathbb{Z}_2$-grading on $\mathcal{S}$. In fact, we have.

<u>Prop. 4.6</u>: Equipped w/ the above $\mathbb{Z}_2$-grading $\mathcal{S}$ is a $\mathbb{Z}_2$-graded module over $Cl_{\mathbb{C}}(\mathbb{R}^n)$.

<u>Proof</u>: As n even, $\Gamma c(v) = -c(v)\Gamma$ $\forall v \in \mathbb{C}^n$. It then follows that Γ commutes with the elements of $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$ and anticommutes with those of $Cl_{\mathbb{C}}^-(\mathbb{R}^n)$. Therefore, if $a \in Cl^+(\mathbb{R}^n)$ and $\sigma \in \mathcal{S}^\pm$, then

$$\rho(\Gamma)(\rho(a)\sigma) = \rho(\Gamma a)\sigma = \rho(a\Gamma)\sigma = \rho(a)(\rho(\Gamma)a) = \pm \rho(a)\sigma.$$

Likewise, if $a \in Cl_{\mathbb{C}}^-(\mathbb{R}^n)$ and $\sigma \in \mathcal{S}^\pm$, then

$$\rho(\Gamma)(\rho(a)\sigma) = \rho(\Gamma a)\sigma = -\rho(a\Gamma)\sigma = -\rho(a)(\rho(\Gamma)a) = \mp \rho(a)\sigma$$

This shows that the action of $Cl_{\mathbb{C}}^+(\mathbb{R}^n)$ preserves $\mathcal{S}^\pm$, while that of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ maps $\mathcal{S}^\pm$ to $\mathcal{S}^\mp$. This proves that $\mathcal{S}$ is a $\mathbb{Z}_2$-graded module over $Cl_{\mathbb{C}}(\mathbb{R}^n)$.

By construction $\mathcal{S} = \Lambda^{0,*}\mathbb{C}^n = \bigoplus_{q=0}^{m} \Lambda^{0,q}\mathbb{C}^n$ (where $m = \frac{n}{2}$). Therefore, we have another $\mathbb{Z}_2$-grading,

$$\mathcal{S} = \Lambda^{0,ev}\mathbb{C}^n \oplus \Lambda^{0,odd}\mathbb{C}^n,$$

$$\Lambda^{0,ev}\mathbb{C}^n = \bigoplus_{\substack{q=0 \\ q \text{ even}}}^{m} \Lambda^{0,q}\mathbb{C}^n, \qquad \Lambda^{0,odd}\mathbb{C}^n = \bigoplus_{\substack{q=0 \\ q \text{ odd}}}^{m} \Lambda^{0,q}\mathbb{C}^n.$$

As it turns, we actually have only one $\mathbb{Z}_2$-grading. Namely, we have.

<u>Lemma 4.7</u>: $\mathcal{S}^+ = \Lambda^{0,ev}\mathbb{C}^n$ and $\mathcal{S}^- = \Lambda^{0,odd}\mathbb{C}^n$.

<u>Proof</u>: What we need to show is that $\rho(\Gamma) = 1$ on $\Lambda^{0,ev}\mathbb{C}^n$ and $\rho(\Gamma) = -1$ on $\Lambda^{0,odd}$, or equivalently,

$$\rho(\Gamma)\xi = (-1)^q \xi \quad \forall \xi \in \Lambda^{0,q}\mathbb{C}^n \quad \forall q \in \{0, \ldots, m\}.$$

For $j=1,\dots,m$ let π_j be the orthogonal projection onto $\operatorname{im}(\mathcal{E}(z^{\bar{j}})) = z^{\bar{j}} \wedge \Lambda^{0,*}\mathbb{C}^m$. As $\operatorname{im}\mathcal{E}(z^{\bar{j}})$ is spanned by $\{z^{\bar{I}}\}_{I \ni j}$, we see that $\ker \pi_j$ is spanned by $\{z^{\bar{I}}\}_{I \not\ni j}$. In addition, define

$$F = \prod_{j=1}^{m} (1 - 2\pi_j).$$

Notice that

$$(1-2\pi_j) z^{\bar{I}} = \begin{cases} -z^{\bar{I}} & \text{if } j \in I, \\ z^{\bar{I}} & \text{if } j \notin I. \end{cases}$$

Therefore,

$$F z^{\bar{I}} = (-1)^{|I|} z^{\bar{I}},$$

that is, for all $q = 0,\dots,m$

$$F|_{\Lambda^{0,q}\mathbb{C}^m} = (-1)^q.$$

Observe also that if $j \in I$, then

$$\iota(z^{j})\mathcal{E}(z^{\bar{j}}) z^{\bar{I}} = 0,$$

but if $j \notin I$, then

$$\iota(z^{j})\mathcal{E}(z^{\bar{j}}) z^{\bar{I}} = \iota(z^{j})(z^{\bar{j}} \wedge z^{\bar{I}}) = z^{\bar{I}}.$$

Therefore, we see that $\iota(z^j)\mathcal{E}(z^{\bar{j}})$ is the orthogonal projection onto $\operatorname{span}\{z^{\bar{I}}\}_{I \not\ni j} = \ker \pi_j$, that is, $\iota(z^j)\mathcal{E}(z^{\bar{j}}) = 1 - \pi_j$. Thus,

$$1 - 2\pi_j = 2\iota(z^j)\mathcal{E}(z^{\bar{j}}) - 1 = -\rho(c(z^j))\rho(c(z^{\bar{j}})) - 1 = -\rho\left(c(z^j)c(z^{\bar{j}}) + 1\right).$$

Observe further that

$$\begin{aligned} c(z^j)c(z^{\bar{j}}) &= \frac{1}{2}\{c(e^{2j}) + i c(e^{2j+1})\}\{c(e^{2j}) - i c(e^{2j+1})\} \\ &= \frac{1}{2}\{c(e^{2j})^2 + c(e^{2j+1})^2 - i c(e^{2j})c(e^{2j+1}) + i c(e^{2j+1})c(e^{2j})\} \\ &= -(1 + i c(e^{2j})c(e^{2j+1})) \end{aligned}$$

Thus,

$$\rho^{-1}(1-2\pi_j) = -(c(z^j)c(z^{\bar{j}}) + 1) = i c(e^{2j})c(e^{2j+1}),$$

and hence

$$\rho^{-1}(F) = \prod_{j=1}^{q} i c(e^{2j})c(e^{2j+1}) = i^m c(e^1)c(e^2)\cdots c(e^{2m-1})c(e^{2m}) = \Gamma.$$

It then follows that

$$\rho(\Gamma)|_{\Lambda^{0,q}} = F|_{\Lambda^{0,q}} = (-1)^q \qquad \text{for all } q \in \{0,\dots,m\}$$

This completes the proof. $\square$

We shall now give a formula for the supertrace of an endomorphism $T \in \mathrm{End}(\$)$. Recall that if using the splitting $\$ = \$^+ \oplus \$^-$ we write T as $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then
$$\mathrm{Str}_\$(T) = \mathrm{Tr}_{\$^+}(A) - \mathrm{Tr}_{\$^-}(D)$$

We also observe that
$$\rho(\Gamma) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right),$$
so we see that
$$\mathrm{Str}_\$(T) = \mathrm{Tr}_\$(\rho(\Gamma)T).$$

Notice further that the fact that $\$$ is a $\mathbb{Z}_2$-graded module exactly means that the spinor representation $\rho: Cl_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}(\$)$ gives rise to an isomorphism of $\mathbb{Z}_2$-graded algebras. Thus, at the level of supercommutators,
$$[\rho(a), \rho(b)]' = \rho([a,b]') \qquad \forall a, b \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$
In addition, as $\rho: Cl_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}(\$)$ is an isomorphism and $\sigma: Cl_{\mathbb{C}}(\mathbb{R}^n) \to \Lambda^* \mathbb{C}^n$, we get a linear isomorphism,
$$\sigma \circ \rho^{-1}: \mathrm{End}(\$) \to \Lambda^* \mathbb{C}^n$$
This map is sometimes referred to as the <u>symbol map</u>.

In the sequel, we denote by $\omega_{\mathbb{R}^n}$ the volume form of $\mathbb{R}^n$, i.e.,
$$\omega_{\mathbb{R}^n} = e^1 \wedge \cdots \wedge e^n, \qquad \{e^j\} \text{ oriented orthonormal basis of } \mathbb{R}^n.$$

<u>Proposition 4.8</u>: For all $T \in \mathrm{End}(\$)$,
$$\mathrm{Str}_\$(T) = (-2i)^{n/2} \langle \omega_{\mathbb{R}^n}, \sigma \circ \rho^{-1}(T)^{(n)} \rangle,$$
where $\sigma \circ \rho^{-1}(T)^{(n)}$ denotes the component of $\sigma \circ \rho^{-1}(T)$ in $\Lambda^n \mathbb{C}^n$.

<u>Proof</u>: Let $\eta \in \Lambda^* \mathbb{C}^n$ and let us write $\eta = \eta^{(n)} + \xi$ with $\eta^{(n)} \in \Lambda^n \mathbb{C}^n$ and $\xi \in \bigoplus_{j=0}^{n-1} \Lambda^j \mathbb{C}^n$. By Lemma 3.7 $c\left(\bigoplus_{j=0}^{n-1} \Lambda^j \mathbb{C}^n\right)$ is contained in the supercommutator space of $Cl_{\mathbb{C}}(\mathbb{R}^n)$, so under ρ it is mapped to the supercommutator space of $\mathrm{End}(\$)$. As the supertrace vanishes on supercommutators (cf. Lemma 1.1), we deduce that
$$\mathrm{Str}_\$[\rho(c(\xi))] = 0$$
and hence
$$\mathrm{Str}_\$[\rho(c(\eta))] = \mathrm{Str}_\$[\rho(c(\eta^{(n)}))].$$

As $\eta^{(n)} = \langle \omega_{\mathbb{R}^n}, \eta^{(n)} \rangle$ and $\Gamma = i^{n/2} c(\omega_{\mathbb{R}^n})$, we see that
$$c(\eta^{(n)}) = \langle \omega_{\mathbb{R}^n}, \eta^{(n)} \rangle c(\omega_{\mathbb{R}^n}) = (-i)^{n/2} \langle \omega, \eta^{(n)} \rangle \Gamma,$$
and hence
$$\mathrm{Str}_{\mathcal{S}}\, \rho(c(\eta)) = (-i)^{n/2} \langle \omega, \eta^{(n)} \rangle \mathrm{Str}_{\mathcal{S}}(\rho(\Gamma)).$$
As $\rho(\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that
$$\mathrm{Str}_{\mathcal{S}}\, \rho(\Gamma) = \mathrm{Tr}_{\mathcal{S}^+} \mathrm{id}_{\mathcal{S}^+} - \mathrm{Tr}_{\mathcal{S}^-}(-\mathrm{id}_{\mathcal{S}^-}) = \mathrm{Tr}_{\mathcal{S}^+} \mathrm{id}_{\mathcal{S}^+} + \mathrm{Tr}\, \mathrm{id}_{\mathcal{S}^-} = \dim \mathcal{S}^+ + \dim \mathcal{S}^- = \dim \mathcal{S} = 2^{n/2}.$$
Therefore, we see that
$$\mathrm{Str}_{\mathcal{S}}[\rho(c(\eta))] = (-2i)^{n/2} \langle \omega, \eta^{(n)} \rangle \quad \forall \eta \in \Lambda^* \mathbb{C}^n.$$
Applying this to $\eta = \sigma \circ \rho^{-1}(T)$ with $T \in \mathrm{End}(\mathcal{S})$ then gives the proposition. □

5. Modules over $C\ell(\mathbb{R}^n)$:

In the section, we assume n even and we look at $\mathbb{Z}_2$-graded modules over $C\ell_{\mathbb{C}}(\mathbb{R}^n)$.

Thanks to Prop. 4.6 we know that $\mathcal{S}$ is $\mathbb{Z}_2$-graded module over $C\ell_{\mathbb{C}}(\mathbb{R}^n)$. In the sequel, we shall simply denote by $C\ell_{\mathbb{C}}(\mathbb{R}^n) \times \mathcal{S} \ni (a, \sigma) \to$ ~~$a\sigma$ the action of $C\ell_{\mathbb{C}}(\mathbb{R}^n)$~~ on $\mathcal{S}$, i.e., $a.\sigma = \rho(a)\sigma$.

Moreover by Prop. 4.5 the spinor representation is a $*$-representation. As mentioned earlier this implies that the grading operator $\rho(\Gamma)$ is selfadjoint and its eigenspaces $\mathcal{S}^{\pm} = \ker(\rho(\Gamma) \mp 1)$ are orthogonal complements of each other.

Def.: Let E be a $C\ell_{\mathbb{C}}(\mathbb{R}^n)$-module, i.e., there is a representation $\pi : C\ell_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}_{\mathbb{C}}(E)$. Then we say that the module E is <u>selfadjoint</u> when it is endowed with a Hermitian inner product with respect to which π is a $*$-representation, i.e., $\pi(a^*) = \pi(a)^* \ \forall a \in C\ell_{\mathbb{C}}(\mathbb{R}^n)$.

Def.: Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded $C\ell_{\mathbb{C}}(\mathbb{R}^n)$-module associated to the ($\mathbb{Z}_2$-graded) representation $\pi : C\ell_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}(E)$. We say that E is a selfadjoint $\mathbb{Z}_2$-graded module over $C\ell_{\mathbb{C}}(\mathbb{R}^n)$ when E is equipped with a Hermitian product with respect to which

- The $\mathbb{Z}_2$-grading $E = E_+ \oplus E_-$ is an orthogonal decomposition.
- The representation π is a $*$-representation.

As explained above, the space of spinors $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is an example of ~~a~~ selfadjoint $\mathbb{Z}_2$-graded $C\ell_{\mathbb{C}}(\mathbb{R}^n)$-module. Other examples are obtained as follows.

Let W be a Hermitian space. Then $\mathcal{S} \otimes W$ is a $C\ell_{\mathbb{C}}(\mathbb{R}^n)$-module with respect to the action of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ defined by

$$a.(\sigma\otimes\omega) := (a\sigma)\otimes\omega \qquad \forall(\sigma,\omega)\in \$\times W \quad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

That is, the corresponding representation is $\rho\otimes 1_W : Cl_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow \mathrm{End}(\$\otimes W)$.
If we denote by $\langle\cdot,\cdot\rangle_\$$ and $\langle\cdot,\cdot\rangle_W$ the respective Hermitian inner products of $\$$ and W, then we ~~can~~ endow $\$\otimes W$ with the Hermitian inner product defined by

$$\langle \sigma_1\otimes\omega_1, \sigma_2\otimes\omega_2\rangle_{\$\otimes W} = \langle\sigma_1,\sigma_2\rangle\langle\omega_1,\omega_2\rangle \qquad \forall \sigma_j\in\$ \ \forall \omega_j\in W.$$

Let $a\in Cl_{\mathbb{C}}(\mathbb{R}^n)$. Then, for all $\sigma_j\in\$$ and $\omega_j\in W$, $j=1,2$, we have

$$\begin{aligned}
\langle(\rho\otimes 1_\$)(a)^*(\sigma_1\otimes\omega_1), \sigma_2\otimes\omega_2\rangle_{\$\otimes W} &= \langle\sigma_1\otimes\omega_1, (\rho\otimes 1_\$)(\sigma_2\otimes\omega_2)\rangle_{\$\otimes W}\\
&= \langle\sigma_1\otimes\omega_1, (\rho(a)\sigma_2)\otimes\omega_2\rangle_{\$\otimes W}\\
&= \langle\sigma_1, \rho(a)\sigma_2\rangle_\$ \langle\omega_1,\omega_2\rangle_W\\
&= \langle\rho(a)^*\sigma_1,\sigma_2\rangle_\$\langle\omega_1,\omega_2\rangle_W\\
&= \langle\rho(a^*)\sigma_1,\sigma_2\rangle_\$\langle\omega_1,\omega_2\rangle_W\\
&= \langle(\rho\otimes 1_\$)(a^*)(\sigma_1\otimes\omega_1), \sigma_2\otimes\omega_2\rangle_{\$\otimes W}
\end{aligned}$$

This proves that

$$(\rho\otimes 1_\$)(a)^* = (\rho\otimes 1_\$)(a^*) \qquad \forall a\in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

That is, $\rho\otimes 1_\$$ is a $*$-representation and $\$\otimes W$ is a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.
Suppose now that W has an orthogonal $\mathbb{Z}_2$-grading $W = W^+\oplus W^-$. Then $\$\otimes W$ is equipped with a $\mathbb{Z}_2$-grading given in Section 1, i.e., $\$\otimes W = (\$\otimes W)^+\oplus(\$\otimes W)^-$, with

$$(\$\otimes W)^+ = (\$^+\otimes W^+)\oplus(\$^-\otimes W^-),$$
$$(\$\otimes W)^- = (\$^+\otimes W^-)\oplus(\$^-\otimes W^+).$$

Then $(\$\otimes W)^+$ and $(\$\otimes W)^-$ are orthogonal subspaces. Moreover,

$$Cl_{\mathbb{C}}^{\pm}(\mathbb{R}^n)(\$\otimes W) = (Cl_{\mathbb{C}}^{\pm}(\mathbb{R}^n)\$)\otimes W,$$

and hence

$$\begin{aligned}
Cl_{\mathbb{C}}^+(\mathbb{R}^n)(\$\otimes W)^{\pm} &= Cl_{\mathbb{C}}^+(\mathbb{R}^n)(\$^{\pm}\otimes W^+\oplus\$^{\mp}\otimes W^-)\\
&\subset \{Cl_{\mathbb{C}}^+(\mathbb{R}^n)\$^{\pm})\otimes W^+\}\oplus\{(Cl_{\mathbb{C}}^+(\mathbb{R}^n)\$^{\mp})\otimes W^-\}\\
&\subset (\$^{\pm}\otimes W^+)\oplus(\$^{\pm}\otimes W^-) = (\$\otimes W)^{\pm}.
\end{aligned}$$

Similarly,

$$Cl_{\mathbb{C}}^-(\mathbb{R}^n)(\$\otimes W)^{\pm}\subset(\$\otimes W)^{\mp}.$$

Therefore, $\$\otimes W$ is a selfadjoint $\mathbb{Z}_2$-graded $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.
We shall the module $\$\otimes W$ a <u>twisted $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module</u>. As we shall now see any $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module (of finite dimension) is isomorphic to a twisted module.

In the sequel by $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module we will exclusively mean a finite dimensional module.

Lemma 5.1: Let E be a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.

(i) We always can endow E with a Hermitian inner product with respect to which E becomes a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.

(ii) If E is $\mathbb{Z}_2$-graded, then we even can choose the inner product so that E is a selfadjoint $\mathbb{Z}_2$-graded $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.

Proof: Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ and define:

$$G = \{\pm 1\} \cup \{\pm c(e^{i_1}) \cdots c(e^{i_\ell});\ 1 \leq i_1 < \cdots < i_\ell \leq n\}.$$

Then G spans $Cl_{\mathbb{C}}(\mathbb{R}^n)$. Moreover, the Clifford relations,

$$c(e^i)^2 = -1, \quad c(e^i)c(e^j) = -c(e^j)c(e^i), \quad i \neq j,$$

imply that G is actually a finite subgroup of $Cl_{\mathbb{C}}(\mathbb{R}^n)$.

Let $\langle \cdot, \cdot \rangle'_E$ be a Hermitian inner product on E, and define a new Hermitian inner product $\langle \cdot, \cdot \rangle_E$ on E by

$$\langle \xi, \eta \rangle_E = \frac{1}{|G|} \sum_{\gamma \in G} \langle \gamma\xi, \gamma\eta \rangle'_E.$$

Then $\langle \cdot, \cdot \rangle_E$ is a G-invariant meaning that $\pi|_G : G \to \mathrm{End}(E)$ is a unitary representation. In particular, for $j = 1, \ldots, n$.

$$\pi(c(e^j))^* = \pi(c(e^j))^{-1}.$$

As $c(e^j)^2 = -1$ and $c(e^j)^* = -c(e^j)$ (cf. Lemma 3.1) we see that

$$\pi(c(e^j))^2 = \pi(c(e^j)^2) = \pi(-1) = -1,$$

and hence

$$\pi(c(e^j))^{-1} = -\pi(c(e^j)) = \pi(-c(e^j)) = \pi(c(e^j)^*).$$

Thus,

$$\pi(c(e^j))^* = \pi(c(e^j)^*) \quad \forall j \in \{1, \ldots, n\}.$$

We then can argue as in the proof of Prop. 4.5 to prove that

$$\pi(a)^* = \pi(a^*) \quad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$

That is, π is a $*$-representation and E is a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module.

Suppose now that E is a $\mathbb{Z}_2$-graded module with $\mathbb{Z}_2$-grading $E = E^+ \oplus E^-$ and let us choose the inner product $\langle \cdot, \cdot \rangle'_E$ so that E^+ and E^- are orthogonal subspaces. Let $(\xi, \eta) \in E^+ \times E^-$ and let $\gamma \in G$. If $\gamma \in Cl^+(\mathbb{R}^n)$, then $(\gamma\xi, \gamma\eta) \in E^+ \times E^-$, and hence

$$\langle \gamma\xi, \gamma\eta \rangle'_E = 0.$$

If $\gamma \in Cl_{\mathbb{C}}^{-}(\mathbb{R}^n)$, then $(\gamma\xi, \gamma\eta) \in E^{-} \times E^{+}$, and so
$$\langle \gamma\xi, \gamma\eta \rangle'_E = 0.$$
It then follows, that
$$\langle \xi, \eta \rangle_E = \frac{1}{|G|} \sum_{\gamma \in G} \langle \gamma\xi, \gamma\eta \rangle'_E = 0.$$
This proves that E^{+} and E^{-} are orthogonal subspaces with respect to the inner product $\langle \cdot, \cdot \rangle_E$. As π is a $*$-representation with respect to this inner product, we see that E is a selfadjoint $\mathbb{Z}_2$-graded $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module. The lemma is proved. □

<u>Rmk:</u> As it follows from the proofs of Prop. 4.5 and of the above lemma, if E is a Hermitian vector space, then a representation $\pi : Cl_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}(E)$ is a $*$-representation if and only if,
$$\pi(c(v))^* = -\pi(c(v)) \quad \forall v \in \mathbb{R}^n.$$
Moreover, if π is a $*$-representation, then each endomorphism $\pi(c(v))$ is unitary and skew-adjoint.

In the sequel, for any given $k \in \mathbb{N}$, we endow $\$^k$ with the structure of selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module given by the diagonal action of $Cl_{\mathbb{C}}(\mathbb{R}^n)$, namely,
$$a.(\sigma_1, \ldots, \sigma_k) := (a\sigma_1, \ldots, a\sigma_k) \quad \forall \sigma_j \in \$ \quad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n).$$
Notice that $\$^k$ is a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module since for each $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$,
$$(\rho(a) \oplus \cdots \oplus \rho(a))^* = \rho(a)^* \oplus \cdots \oplus \rho(a)^* = \rho(a^*) \oplus \cdots \oplus \rho(a^*)$$
In this structure of selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module is equivalent to that of $\$ \otimes \mathbb{C}^k$ under the unitary isomorphism $\alpha : \$ \otimes \mathbb{C}^k \to \$^k$ defined by
$$\alpha(\sigma \otimes \lambda) = (\lambda_1 \sigma, \ldots, \lambda_k \sigma) \quad \forall \sigma \in \$ \quad \forall \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k.$$

<u>Lemma 5.2:</u> Let E be a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module. Then for some $k \in \mathbb{N}$ there is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism $E \simeq \$^k$.

<u>Proof:</u> Let $\pi : Cl_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}(E)$ be the representation that defines the module structure of E. This a $*$-representation since E is a selfadjoint $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module. Therefore, we get a $*$-morphism,
$$\pi \circ \rho^{-1} : \mathrm{End}(\$) \to \mathrm{End}(E).$$

For $\sigma_1, \sigma_2 \in \$$ let us denote by $\sigma_2 \otimes \sigma_1^*$ the endomorphism of $\$$ defined by
$$(\sigma_1 \otimes \sigma_2^*)\sigma = \langle \sigma_2, \sigma \rangle \sigma_1 \quad \forall \sigma \in \$.$$
Notice that
$$(\sigma_1 \otimes \sigma_2^*)(\sigma_3 \otimes \sigma_4^*) = \langle \sigma_2, \sigma_3 \rangle \sigma_1 \otimes \sigma_4^*,$$
$$(\sigma_1 \otimes \sigma_2^*)^* = \sigma_2 \otimes \sigma_1^*.$$
Therefore, if $\{\sigma^1, \ldots, \sigma^\ell\}$ is an orthonormal basis of $\$$, then each endomorphism $\sigma_j \otimes \sigma_j^*$ is in fact the orthogonal projection onto $\mathbb{C}\sigma_1$, and hence
$$\sum_{i=1}^{\ell} \sigma^i \otimes (\sigma^i)^* = 1.$$
Thus,
$$\sum_{j=1}^{\ell} (\pi \circ \bar{\rho}^{-1})(\sigma^j \otimes (\sigma^j)^*) = \pi \circ \bar{\rho}^{-1}(1) = 1.$$
This implies that there exists j_0 s.t. $\pi \circ \bar{\rho}^{-1}(\sigma^{j_0} \otimes (\sigma^{j_0})^*) \neq 0$.

Set $\sigma_0 = \sigma^{j_0}$. As $(\sigma_0 \otimes \sigma_0^*)^* = (\sigma_0 \otimes \sigma_0^*)^2 = \sigma_0 \otimes \sigma_0^*$ and $\pi \circ \bar{\rho}^{-1}$ is a $*$-isomorphism we see that $\Pi := \pi \circ \bar{\rho}^{-1}(\sigma_0 \otimes \sigma_0^*)$ satisfies $\Pi^* = \Pi^2 = \Pi$, that is, Π is an orthogonal projection.

As $\Pi = \pi \circ \bar{\rho}^{-1}(\sigma_0 \otimes \sigma_0^*) \neq 0$ we can find a unit vector $\xi \in E$ contained in $\operatorname{im} \Pi$, i.e., $\Pi \xi = \xi$. We then define a linear map $\varphi : \$ \rightarrow E$ by
$$\varphi(\sigma) = \pi \circ \bar{\rho}^{-1}(\sigma \otimes \sigma_0^*)\xi.$$
Let $a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$. Then, as $T(\sigma \otimes \sigma_0^*) = (T\sigma) \otimes \sigma_0^* \;\; \forall T \in \operatorname{End}(\$)$, we see that
$$\begin{aligned}
\varphi(a\sigma) &= \pi \circ \bar{\rho}^{-1}((\rho(a)\sigma) \otimes \sigma_0^*) = \pi \circ \bar{\rho}^{-1}(\rho(a)(\sigma \otimes \sigma_0^*))\xi \\
&= \pi \circ \bar{\rho}^{-1}(\rho(a))\, \pi \circ \bar{\rho}^{-1}(\sigma \otimes \sigma_0^*)\xi \\
&= \pi(a)\, \varphi(\sigma) \\
&= a\varphi(\sigma).
\end{aligned}$$
Thus φ is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant map. Therefore, $\ker \varphi$ is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-invariant subspace of $\$$. Observe that $\ker \varphi \neq \$$, since
$$\varphi(\sigma_0) = \pi \circ \bar{\rho}^{-1}(\sigma_0 \otimes \sigma_0^*)\xi = \Pi \xi = \xi \neq 0.$$
As $\$$ is an irreducible module (cf. Thm. 4.2), we deduce that $\ker \varphi = \{0\}$, i.e., φ is injective.

Notice that $E_1 := \operatorname{im} \varphi$ is $Cl_{\mathbb{C}}(\mathbb{R}^n)$-invariant subspace of E, and hence is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module. Thus, φ gives rise to an isomorphism of $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module from $\$$ onto $\$$,

This isomorphism is isometric. Indeed, for $\sigma_1, \sigma_2 \in \$$,
$$\begin{aligned}
\langle \varphi(\sigma_1), \varphi(\sigma_2) \rangle_E &= \langle \pi \circ \bar{\rho}^{-1}(\sigma_1 \otimes \sigma_0^*)\xi, \pi \circ \bar{\rho}^{-1}(\sigma_2 \otimes \sigma_0^*)\xi \rangle_E \\
&= \langle (\pi \circ \bar{\rho}^{-1}(\sigma_2 \otimes \sigma_0^*))^* \pi \circ \bar{\rho}^{-1}(\sigma_1 \otimes \sigma_0^*)\xi, \xi \rangle_E
\end{aligned}$$

Observe that

$$\left(\pi\circ\rho^{-1}(\sigma_2\otimes\sigma_0^*)\right)^*\pi\circ\rho^{-1}(\sigma_1\otimes\sigma_0^*)=\pi\circ\rho^{-1}\left\{(\sigma_2\otimes\sigma_0^*)^*\sigma_1\otimes\sigma_0^*\right\}$$

$$=\pi\circ\rho^{-1}\left\{(\sigma_0\otimes\sigma_2^*)(\sigma_1\otimes\sigma_0^*)\right\}$$

$$=\pi\circ\rho^{-1}\left\{\langle\sigma_2,\sigma_1\rangle_{\mathcal{S}}\,\sigma_0\otimes\sigma_0^*\right\}$$

$$=\langle\sigma_2,\sigma_1\rangle_{\mathcal{S}}\,\Pi.$$

Thus,

$$\langle\varphi(\sigma_1),\varphi(\sigma_2)\rangle_E=\langle\sigma_2,\sigma_1\rangle_{\mathcal{S}}\,\langle\Pi\xi,\xi\rangle_E=\langle\sigma_2,\sigma_1\rangle_{\mathcal{S}}\,\langle\xi,\xi\rangle_E=\langle\sigma_2,\sigma_1\rangle_{\mathcal{S}},$$

which shows that φ is an isometric isomorphism from $\mathcal{S}$ onto E_1.

If $\dim E_1=\dim E$, then $E=E_1$ and we are done; φ is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism from $\mathcal{S}$ onto E.

Suppose now that $E_1\subsetneq E$. Then $E_1^\perp$ is a non-trivial $Cl_{\mathbb{C}}(\mathbb{R}^n)$-invariant subspace of E. More precisely, if $\xi\in E_1^\perp$ and $a\in Cl_{\mathbb{C}}(\mathbb{R}^n)$, then using the fact that π is a $*$-representation we see that, for all $\eta\in E_1$,

$$\langle a\xi,\eta\rangle=\langle\pi(a)\xi,\eta\rangle=\langle\xi,\pi(a)^*\eta\rangle=\langle\xi,\pi(a^*)\eta\rangle=0,$$

because $\pi(a^*)\eta\in E_1$. We thus get an orthogonal decomposition into non-trivial $Cl_{\mathbb{C}}(\mathbb{R}^n)$-invariant subspaces,

$$E=E_1\oplus E_1^\perp,$$

in such a way that there is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism $\varphi_1:\mathcal{S}\rightarrow E_1$.

By repeating the above process we can successively construct orthogonal $Cl_{\mathbb{C}}(\mathbb{R}^n)$-invariant subspaces $E_1,\dots,E_\ell,F_\ell$, such that:

- For each $j=1,\dots,\ell$ there is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism $\varphi_j:\mathcal{S}\rightarrow E_j$.
- We have an orthogonal decomposition:

$$E=E_1\oplus\cdots\oplus E_\ell\oplus F_\ell.$$

If we repeat this process enough times we can obtain that $\dim F_\ell\leq\dim\mathcal{S}$. Then, then as shown above we can construct a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric linear map $\psi:\mathcal{S}\rightarrow F_\ell$. As $\dim F_\ell\leq\dim\mathcal{S}$, we see that $\dim F_\ell=\dim\mathcal{S}$ and φ is an isomorphism. Setting $E_{\ell+1}=F_\ell$ and $\varphi_{\ell+1}=\psi$ we have the orthogonal decomposition

$$E=E_1\oplus\cdots\oplus E_{\ell+1},$$

and we can define a linear map $\mathcal{S}^{\ell+1}\rightarrow E$ by

$$\varphi(\sigma):=\varphi_1(\sigma_1)+\cdots+\varphi_{\ell+1}(\sigma_{\ell+1})\qquad\forall\,\sigma=(\sigma_j)_{1\leq j\leq\ell+1}\in\mathcal{S}^{\ell+1}.$$

This map is onto and $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant since so are all the φ_j. Moreover, as the decomposition $E = E_1 \oplus \cdots \oplus E_{\ell+1}$ is an orthogonal decomposition and all the φ_j are isometric, we see that, for all $\sigma = (\sigma_j) \in \$^{\ell}$,

$$\|\varphi(\sigma)\|_E^2 = \sum \|\varphi_j(\sigma_j)\|_E^2 = \sum \|\sigma_j\|_{\$}^2 = \|\sigma\|_{\$^{\ell}}^2,$$

so that φ is isometric. This implies that φ is one-to-one, so we deduce that φ is a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism from $\$^{\ell}$ onto E. The lemma is proved. $\square$

<u>Prop. 5.3</u>: Let $E = E^+ \oplus E^-$ be a selfadjoint $\mathbb{Z}_2$-graded module over $Cl_{\mathbb{C}}(\mathbb{R}^n)$. Let us equip $W = \mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, V)$ with the $\mathbb{Z}_2$-grading defined in Section 1 and endow $\$ \otimes W$ with the structure of $\mathbb{Z}_2$-graded $Cl_{\mathbb{C}}(\mathbb{R}^n)$-module defined on page 29. Let $\beta : S \otimes W \to E$ be the linear map defined by

$$\beta(\sigma \otimes w) := w(\sigma) \quad \forall (\sigma, w) \in \$ \times W$$

Then β is a $\mathbb{Z}_2$-graded $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant linear isomorphism.

<u>Proof.</u>: By Lemma 5.2 for some $\ell \in \mathbb{N}$ we have a $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant isometric isomorphism $\varphi : E \to \$^{\ell}$

$$\varphi(\xi) = (\varphi_1(\xi), \ldots, \varphi_{\ell}(\xi)) \quad \forall \xi \in E.$$

This naturally gives rise to the linear isomorphism $\varphi_{\#} : W = \mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, E) \to \mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, \$)^{\ell}$ given by

$$\varphi_{\#}(w) = (\varphi_1 \circ w, \ldots, \varphi_{\ell} \circ w).$$

Let $T \in \mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, \$)$. The fact that T is $Cl_{\mathbb{C}}(\mathbb{R}^n)$-equivariant here means that

$$T\rho(a) = \rho(a) T \quad \forall a \in Cl_{\mathbb{C}}(\mathbb{R}^n)$$

As $\rho(Cl_{\mathbb{C}}(\mathbb{R}^n)) = \mathrm{End}(\$)$ this implies that T commutes with all the elements of $\mathrm{End}(\$)$, and hence must be a scalar multiple of the identity. Thus $\mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, \$) = \mathbb{C}.\mathrm{id}_{\$}$ and, by fixing a unit element $\sigma_0 \in \$$, the map

$$\gamma : \mathrm{Hom}_{Cl_{\mathbb{C}}(\mathbb{R}^n)}(\$, \$)^{\ell} \ni (T_j)_{1 \leq j \leq \ell} \longrightarrow (\langle \sigma_0, T_j \sigma_0 \rangle)_{1 \leq j \leq \ell} \in \mathbb{C}^{\ell}$$

is a linear isomorphism from $\mathrm{Hom}(\$, \$)^{\ell}$ onto $\mathbb{C}^{\ell}$. Composing it with $\varphi_{\#}$ yields the linear isomorphism,

$$\tilde{\varphi}_{\#} : W \longrightarrow \mathbb{C}^{\ell}$$
$$w \longrightarrow (\langle \sigma_0, \varphi_j \circ w(\sigma_0) \rangle)_{1 \leq j \leq \ell}.$$

Denote by $\alpha : \$ \otimes \mathbb{C}^{\ell} \to \$^{\ell}$ the linear isomorphism,

$$\alpha(\sigma \otimes \lambda) := (\lambda_1 \sigma, \ldots, \lambda_{\ell} \sigma) \quad \forall \sigma \in \$ \quad \forall \lambda = (\lambda_j) \in \mathbb{C}^{\ell}.$$

Let $(\sigma, w) \in \$ \otimes W$. Then

$$\varphi \circ \beta(\sigma \otimes w) = \varphi(w(\sigma)) = (\varphi_1(w(\sigma)), \ldots, \varphi_1(w(\sigma)))$$

As $\varphi_j \circ \omega$ is contained in $\mathrm{Hom}_{Cl_q(\mathbb{R}^n)}(\$,\$)$ it agrees with $\langle \sigma_0, \varphi_j \circ \omega(\sigma_0)\rangle \, id_\$$, and hence

$$\varphi_j(\omega(\sigma)) = \langle \sigma_0, \varphi_j \circ \omega(\sigma_0)\rangle \sigma.$$

Thus,

$$\varphi \circ \beta = (\langle \sigma_0, \varphi_1 \circ \omega(\sigma_0)\rangle \sigma, \ldots, \langle \sigma_0, \varphi_q \circ \omega(\sigma_0)\rangle \sigma)$$

On the other hand,

$$\alpha \circ (1 \otimes \tilde{\varphi}_\#)(\sigma \otimes \omega) = \alpha\left(\sigma \otimes (\langle \sigma_0, \varphi_j \circ \omega(\sigma_0)\rangle)_{1 \leq j \leq q}\right)$$

$$= (\langle \sigma_0, \varphi_j \circ \omega(\sigma_0)\rangle \sigma)_{1 \leq j \leq q}.$$

Therefore we see that we have a commutative diagram,

$$\begin{array}{ccc} \$ \otimes W & \xrightarrow{\beta} & E \\ {\scriptstyle 1 \otimes \tilde{\varphi}} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \$ \otimes \mathbb{C}^q & \xrightarrow{\alpha} & \$^q \end{array}$$

As the vertical maps and the bottom horizontal maps are linear isomorphisms we see that β is a linear isomorphism.

Moreover, β is $Cl_q(\mathbb{R}^n)$-equivariant, since for $a \in Cl_q(\mathbb{R}^n)$ and $(\sigma, \omega) \in \$ \otimes W$, we have

$$\beta(a(\sigma \otimes \omega)) = \beta((a\sigma) \otimes \omega) = \omega(a\sigma) = a\,\omega(\sigma),$$

since ω is $Cl_q(\mathbb{R}^n)$-equivariant.

In addition, recall that by definition of the $\mathbb{Z}_2$-grading of $\$ \otimes W$ and $W = \mathrm{Hom}_{Cl_q(\mathbb{R}^n)}(\$, E)$,

$$\begin{cases} (\$ \otimes W)^+ = (\$^+ \otimes W^+) \oplus (\$^- \otimes W^-), \\ (\$ \otimes W)^- = (\$^+ \otimes W^-) \oplus (\$^- \otimes W^+), \end{cases}$$

and W^+ (resp., W^-) consists of $T \in \mathrm{Hom}_{Cl_q(\mathbb{R}^n)}(\$, E)$ mapping $\$^\pm$ to $E^\pm$ (resp., $E^\mp$).

Now, if $\sigma \in \$^\pm$ and ω is contained in W^+ (resp., W^-), then $\beta(\sigma \otimes \omega) = \omega(\sigma)$ is contained in $E^\pm$ (resp. $E^\mp$). This means that β maps $(\$ \otimes W)^\pm$ to $E^\pm$, that is, β is $\mathbb{Z}_2$-graded.

It remains to check that β is isometric. Let

Let us endow $W = \mathrm{Hom}_{Cl_q(\mathbb{R}^n)}(\$, E)$ with the Hermitian inner product defined by

$$\langle \omega_1, \omega_2 \rangle_W := 2^{-\frac{n}{2}} \mathrm{Tr}_\$ (\omega_1^* \omega_2) \quad \forall \omega_j \in W$$

Notice that if $w_1 \in W^+$ and $w_2 \in W^-$, then $w_2^* \circ w_1$ maps $E^\pm_\bullet$ to $E^\mp$, and hence has a vanishing trace. Thus W^+ and W^- are orthogonal subspaces. Therefore, if we endow $\$\otimes W$ with the Hermitian inner product defined on page 29, then $\$\otimes W$ becomes a selfadjoint $\mathbb{Z}_2$-graded module.

Prop. 5.4: The isomorphism $\beta : \$\otimes W \to E$ of Prop. 5.3 is isometric.

Proof: Let us first observe that in the diagram (5.1) both φ and α are isometric isomorphisms. Therefore, we need only to show that $\tilde{\varphi}_\#$ is an isometric isomorphism from W onto $\mathbb{C}^\ell$.

We endow $V = \mathrm{Hom}_{Cl(n)}(\$,\$)$ with the Hermitian inner product defined by

$$\langle S, T\rangle_V := 2^{-n/2}\mathrm{Tr}_\$(S^* T). \qquad \forall S,T \in \mathrm{Hom}_{Cl(n)}(\$,\$).$$

In fact, as $S = \langle \sigma_0, S\sigma_0\rangle id_\$$ and $T = \langle \sigma_0, T\sigma_0\rangle id_\$$, we see that

$$\langle S, T\rangle_V := 2^{-n/2}\mathrm{Tr}\left(\overline{\langle \sigma_0, S\sigma_0\rangle}\langle \sigma_0, T\sigma_0\rangle id_\$\right)$$
$$= \overline{\langle \sigma_0, S\sigma_0\rangle}\langle \sigma_0, T\sigma_0\rangle$$

It then follows that γ is an isometric isomorphism from $\mathrm{Hom}_{Cl(n)}(\$,\$)^\ell$ onto $\mathbb{C}^\ell$.

Let $w_1, w_2 \in W$. Then

$$2^{n/2}\langle \varphi_\#(w_1), \varphi_\#(w_2)\rangle_{V^\ell} = \sum_{j=1}^{\ell} 2^{n/2}\langle \varphi_j\circ w_1, \varphi_j\circ w_2\rangle_V$$
$$= \sum_{j=1}^{\ell}\mathrm{Tr}_\$(w_1^*\varphi_j^*\varphi_j w_2)$$

$$= \mathrm{Tr}_\$(w_1^* T w_2),$$

where we have $T = \sum_{j=1}^{\ell}\varphi_j^*\varphi_j \in \mathrm{Hom}_{Cl(n)}(E,E)$. In fact, as φ is isometric, for all $\xi\in E$,

$$\langle \xi, \xi\rangle_E = \langle \varphi(\xi), \varphi(\xi)\rangle_{\$^\ell} = \sum \langle \varphi_j(\xi), \varphi_j(\xi)\rangle_{\$^\ell}$$
$$= \sum \langle \varphi_j^*\varphi_j(\xi), \xi\rangle_E = \langle T\xi, \xi\rangle.$$

Since T is selfadjoint, it follows that $T = 1_E$, and hence

$$\langle \varphi_\#(w_1), \varphi_\#(w_2)\rangle_{V^\ell} = 2^{-n/2}\mathrm{Tr}_\$(w_1^* w_2) = \langle w_1, w_2\rangle_W.$$

This shows that $\varphi_\#$ is an isometric isomorphism. Then $1\otimes\tilde{\varphi}_\# = 1\otimes(\gamma\circ\varphi_\#)$ is an isometric isomorphism. This completes the proof. $\square$

Finally, let $E = \$ \otimes W$ be a twisted $\mathbb{Z}_2$-graded module over $C\ell(\mathbb{R}^n)$, where $W = W^+ \oplus W^-$ is a superspace. Let $T \in \mathrm{End}(W)$. Then

$$\mathrm{Str}_{\$\otimes W}(\Gamma T) = \mathrm{Str}_{\$}(\rho(\Gamma))\,\mathrm{Str}_W(T).$$

As shown in the proof of Prop. 4.8, $\mathrm{Str}_{\$}\rho(\Gamma) = 2^{n/2}$, so we see that

$$\mathrm{Str}_{\$\otimes W}(\Gamma T) = 2^{n/2}\,\mathrm{Str}_W(T),$$

that is,

$$\mathrm{Str}_W(T) = 2^{-n/2}\,\mathrm{Str}_{\$\otimes W}(\Gamma T).$$

This formula motivates the following definition.

Def. (Relative Supertrace): Let E be a $\mathbb{Z}_2$-graded $C\ell(\mathbb{R}^n)$-module. Then the relative supertrace on E is the linear form

$$\mathrm{Str}_{E/\$} : \mathrm{Hom}_{C\ell(\mathbb{R}^n)}(E, E) \longrightarrow \mathbb{C}$$

defined by

$$\mathrm{Str}_{E/\$}(T) = 2^{-n/2}\,\mathrm{Str}_E(\Gamma T) \quad \forall T \in \mathrm{Hom}_{C\ell(\mathbb{R}^n)}(E, E).$$

Rmk: Let $S, T \in \mathrm{Hom}_{C\ell(\mathbb{R}^n)}(E, E)$. As Γ commutes with S and T,

$$\Gamma[S,T]' = [\Gamma S, T]',$$

and hence

$$\mathrm{Str}_{E/\$}[S,T]' = 2^{-n/2}\,\mathrm{Str}_E\{\Gamma[S,T]'\} = 2^{-n/2}\,\mathrm{Str}_E[\Gamma S, T]' = 0.$$

Thus $\mathrm{Str}_{E/\$}$ is a supertrace on $\mathrm{Hom}_{C\ell(\mathbb{R}^n)}(E, E)$, in the sense that it vanishes on super-commutators.

6. The Spin Group Spin(n):

Define

$$C\ell^1(\mathbb{R}^n) = c(\mathbb{R}^n) \quad \text{and} \quad C\ell^2(\mathbb{R}^n) = c(\Lambda^2\mathbb{R}^n).$$

Lemma 6.1: (i) $[a, c(v)] \in C\ell^1(\mathbb{R}^n)$ $\forall a \in C\ell^2(\mathbb{R}^n)$ $\forall v \in \mathbb{R}^n$.

(ii) $[a, b] \in C\ell^2(\mathbb{R}^n)$ $\forall a, b \in C\ell^2(\mathbb{R}^n)$.

Proof: Let $\eta \in \Lambda^2(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$. Then by Lemma 3.6,

$$[c(\eta), c(v)] = [c(\eta), c(v)] = 2c(\iota(v)\eta) \in C\ell^1(\mathbb{R}^n).$$

This proves the first part of the Lemma.

Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then, for $i<j$,

$$[c(\eta), c(e^i\wedge e^j)] = [c(\eta), c(e^i)c(e^j)]$$
$$= c(e^i)[c(\eta), c(e^j)] + [c(\eta), c(e^i)]c(e^j)$$
$$= 2c(e^i)c(i(e^j)\eta) + 2c(i(e^i)\eta)c(e^j).$$

Combining this with Lemma 3.3 gives

$$[c(\eta), c(e^i\wedge e^j)] = 2\{c(e^i\wedge i(e^j)\eta) - \langle e^i, i(e^j)\eta\rangle + c(i(e^i)\eta\wedge e^j) - \langle i(e^i)\eta, e^j\rangle\}$$

As lemma 2.3 tells us that $i(e^i)^* = \mathcal{E}(e^i)$ and $i(e^j)^* = \mathcal{E}(e^j)$, we see that

$$\langle e^i, i(e^j)\eta\rangle = \langle i(e^j)^* e^i, \eta\rangle = \langle \mathcal{E}(e^j)e^i, \eta\rangle = \langle e^j\wedge e^i, \eta\rangle$$

Similarly,

$$\langle i(e^i)\eta, e^j\rangle = \langle \eta, e^i\wedge e^j\rangle = -\langle e^i, i(e^j)\eta\rangle.$$

Thus,

$$[c(\eta), c(e^i\wedge e^j)] = 2\,c(e^i\wedge i(e^j)\eta) + c(i(e^i)\eta\wedge e^j) \in Cl^2(\mathbb{R}^n).$$

As the $c(e^i\wedge e^j)$, $i<j$, span $Cl^2(\mathbb{R}^n)$ the 2nd part of the Lemma follows. The Lemma is proved. □

It follows from the 2nd part of Lemma 6.1 that, when equipped with the commutator bracket, $Cl^2(\mathbb{R}^n)$ is a Lie subalgebra of $Cl(\mathbb{R}^n)$. In addition, using the first part of Lemma 6.1, we see that we can define a linear map $\tau: Cl^2(\mathbb{R}^n) \to M_n(\mathbb{R})$ by

$$c(\tau(a).v) = [a, c(v)] \quad \forall a\in Cl^2(\mathbb{R}^n)\ \forall v\in\mathbb{R}^n.$$

Consider the Lie algebra of $SO(n)$, i.e.,

$$so(n) = \{A = (a_{ij});\ a_{ji} = -a_{ij}\ \forall i,j\in\{1,\dots,n\}\}.$$

<u>Prop. 6.2</u> τ gives rise to a Lie algebra isomorphism,

$$\tau: Cl^2(\mathbb{R}^n) \longrightarrow so(n).$$

Moreover, its inverse map is given by

$$\tau^{-1}(A) := \frac{1}{2}\sum_{i<j}\langle Ae^i, e^j\rangle c(e^i)c(e^j) \quad \forall A\in so(n),$$

where $\{e^j\}$ is any orthonormal basis of $\mathbb{R}^n$.

<u>Proof</u>: Let us first show that τ maps to $so(n)$. Let $a\in Cl^2(\mathbb{R}^n)$ and $v, w\in\mathbb{R}^n$. Then

$$-2\langle\tau(a)v, w\rangle = c(\tau(a)v)c(w) + c(w)c(\tau(a)v)$$
$$= [a, c(v)]c(w) + c(w)[a, c(v)].$$

Similarly,

$$-2\langle v, \tau(a)w\rangle = c(v)c(\tau(a)w) + c(\tau(a)w)c(v)$$
$$= c(v)[a, c(w)] + [a, c(w)]c(v).$$

Thus,

$$-2\{\langle \tau(a)v, w\rangle + \langle v, \tau(a)w\rangle\} = [a, c(v)]c(w) + c(v)[a, c(w)] + c(w)[a, c(v)] + [a, c(w)]c(v)$$
$$= [a, c(v)c(w)] + [a, c(w)c(v)]$$
$$= [a, c(v)c(w) + c(w)c(v)]$$
$$= [a, -2\langle v, w\rangle]$$
$$= 0.$$

Therefore, for all $v, w \in \mathbb{R}^n$,

$$\langle \tau(a)v, w\rangle + \langle v, \tau(a)w\rangle = 0.$$

This shows that $\tau(a) \in so(n)$.

Next, let $a, b \in Cl^2(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$. Then

$$c(\tau([a,b])v) = [[a,b], c(v)] = [ab, c(v)] - [ba, c(v)]$$
$$= a[b, c(v)] + [a, c(v)]b - b[a, c(v)] - [b, c(v)]a$$
$$= [a, [b, c(v)]] - [b, [a, c(v)]]$$
$$= [a, c(\tau(b)v)] - [b, c(\tau(a)v)]$$
$$= c(\tau(a).(\tau(b)v)) - c(\tau(b).(\tau(a)v))$$
$$= c([\tau(a), \tau(b)]v).$$

Thus,

$$\tau([a,b]) = [\tau(a), \tau(b)] \qquad \forall a, b \in Cl^2(\mathbb{R}^n),$$

that is, τ is a morphism of Lie algebras.

Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ and for $i,j = 1, \ldots, n$ let us denote by E_{ij} the elementary matrices s.t., for all $k = 1, \ldots, n$,

$$E_{ij}e^k = \delta^{ik}e^j = \langle e^i, e^j\rangle e^k.$$

Then $so(n)$ has basis $\{E_{ij} - E_{ji};\ i<j\}$. Indeed, for any $A \in so(n)$,

$$A = \sum_{i,j=1}^{n} \langle Ae^i, e^j\rangle E_{ij} = \sum_{i<j} \langle Ae^i, e^j\rangle (E_{ij} - E_{ji}).$$

Now, as shown in the proof of Lemma 6.1:

$$[c(e^i)c(e^j), c(e^k)] = [c(e^i \wedge e^j), e^k] = -2c(i(e^k)(e^i \wedge e^j))$$
$$= -2\langle e^k, e^i\rangle c(e^j) + 2\langle e^k, e^j\rangle c(e^i).$$

Thus,

$$\tau(c(e^i)c(e^j))e^k) = \sigma([c(e^i)c(e^j), c(e^k)]) = -2\langle e^j, e^k\rangle e^i + 2\langle e^i, e^k\rangle e^j$$
$$= 2E_{ij}e^k - 2E_{ji}e^k.$$

Therefore, we see that

$$(6.1)\quad \tau(c(e^i)c(e^j)) = 2(E_{ij} - E_{ji}).$$

It then follows that τ maps into $so(n)$. As $\{c(e^i)c(e^j);\, i<j\}$ and $\{E_{ij}-E_{ji};\, i<j\}$ are respective bases of $Cl^2(\mathbb{R}^n)$ and $so(n)$ and have same cardinality, we see that $Cl^2(n)$ and $so(n)$ have same dimension. Therefore, the fact that τ maps into $so(n)$ actually implies that τ is an isomorphism from $Cl^2(\mathbb{R}^n)$ onto $so(n)$.

Finally, (6.1) shows that, for $i<j$,

$$\tau^{-1}(E_{ij} - E_{ji}) = \frac{1}{2} c(e^i)c(e^j).$$

Thus, for all $A \in so(n)$,

$$\tau^{-1}(A) = \sum_{i<j} \langle Ae^i, e^j\rangle \, \tau^{-1}(E_{ij} - E_{ij})$$

$$= \frac{1}{2}\sum_{i<j} \langle Ae^i, e^j\rangle c(e^i)c(e^j).$$

The proof is complete. □

Def. The spin group of $\mathbb{R}^n$, denoted $Spin(n)$, is the Lie group obtained by exponentiating the Lie algebra $Cl^2(\mathbb{R}^n)$ inside $Cl(\mathbb{R}^n)$.

Ex: Let v and w be orthogonal unit vectors in $\mathbb{R}^n$. Then

$$c(v)^2 = c(w)^2 = -1 \quad \text{and} \quad c(v)c(w) = -c(w)c(v).$$

Thus,

$$(c(v)c(w))^2 = c(v)c(w)c(v)c(w) = -c(w)c(v)^2c(w) = c(w)^2 = -1.$$

An easy induction then shows that, for all $\ell \in \mathbb{N}_0$,

$$(c(v)c(w))^{2\ell} = (-1)^\ell \quad \text{and} \quad (c(v)c(w))^{2\ell+1} = (-1)^\ell c(v)c(w).$$

Therefore, for all $t \in \mathbb{R}$,

$$\exp(t\,c(v)c(w)) = \sum_{j=0}^{\infty} \frac{t^j}{j!}(c(v)c(w))^j$$

$$= \sum_{\ell=0}^{\infty} \frac{t^{2\ell}}{(2\ell)!}(c(v)c(w))^{2\ell} + \sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}}{(2\ell+1)!}(c(v)c(w))^{2\ell+1}$$

$$= \sum_{\ell=0}^{\infty} \frac{t^{2\ell}}{(2\ell)!}(-1)^\ell + \sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}}{(2\ell+1)!}(-1)^\ell c(v)c(w).$$

Recognizing the Maclaurin series of $\cos t$ and $\sin t$ we see that
$$\exp(t\,c(v)c(w)) = \cos t + \sin t\, c(v)c(w).$$
In particular, for $t=\pi$ we see that -1 is contained in $\mathrm{Spin}(n)$.

The Lie algebra isomorphism $\tau: Cl^2(\mathbb{R}^n) \to so(n)$ descends to a morphism of Lie groups,
$$\tau: \mathrm{Spin}(n) \longrightarrow SO(n),$$
such that $\tau(\exp(a)) = \exp(\tau(a))$ $\forall a \in Cl^2(\mathbb{R}^n)$.

<u>Lemma 6.3</u>: Let $g \in \mathrm{Spin}(n)$ and let us regard g and g^{-1} as elements of $Cl(\mathbb{R}^n)$. Then
$$g\,c(v)\,g^{-1} = c(\tau(g)v) \quad \forall v \in \mathbb{R}^n.$$

<u>Proof</u>: Let us write $g=\exp(a)$ with $a \in Cl^2(\mathbb{R}^n)$. Then
$$\frac{d}{dt}\{\exp(ta)c(v)\exp(-ta)\} = \exp(ta)[a, c(v)]\exp(-ta) = \exp(ta)c(\tau(a)v)\exp(-ta).$$
A simple induction then shows that, for all $k \in \mathbb{N}$,
$$\frac{d^k}{dt^k}\{\exp(ta)c(v)\exp(-ta)\} = \exp(ta)\,c(\tau(a)^k v)\,\exp(-ta),$$
and hence
$$\frac{d^k}{dt^k}\{\exp(ta)c(v)\exp(-ta)\}\Big|_{t=0} = c(\tau(a)^k v) = \frac{d}{dt}\, c(\exp(t\tau(a))v)\Big|_{t=0}$$
$$= \frac{d}{dt}\, c(\tau(\exp(ta))v)\Big|_{t=0}.$$
As both $\exp(ta)c(v)\exp(-ta)$ and $c(\tau(\exp(ta))v)$ are analytic functions of t we deduce that they agree everywhere. In particular, for $t=1$ we get
$$\exp(a)c(v)\exp(-a) = c(\tau(\exp(a))v),$$
that is,
$$g\,c(v)\,g^{-1} = c(\tau(g)v).$$
The lemma is thus proved. □

<u>Prop. 6.4</u>: The Lie group morphism $\tau: \mathrm{Spin}(n) \to SO(n)$ is a double covering of $SO(n)$, that is, τ is onto and $\ker \tau = \{\pm 1\}$.

<u>Proof</u>: As $\tau: Cl^2(\mathbb{R}^n) \to so(n)$ is an isomorphism and $\exp: so(n) \to SO(n)$ we see that τ maps $\mathrm{Spin}(n)$ onto $SO(n)$. Moreover, as shown in the previous example, -1 is contained in $\mathrm{Spin}(n)$ and by Lemma 6.3, for all $v \in \mathbb{R}^n$,
$$c(\tau(-1)v) = (-1)c(v)(-1)^{-1} = c(v),$$
i.e., $\tau(-1) = 1$. Thus $\{\pm 1\}$ is contained in $\ker \tau$.

Conversely, let $g \in \ker \tau$. Then, by Lemma 6.3, for all $v \in \mathbb{R}^n$,

$$c(v) = c(\tau(g)v) = g\,c(v)\,g^{-1},$$

which implies that $c(v)g = g\,c(v)$. Observe that, in $Cl(\mathbb{R}^n)$, the exponential map leaves $Cl^+(\mathbb{R}^n)$ invariant, so $\mathrm{Spin}(n)$ is contained in $Cl^+(\mathbb{R}^n)$. Therefore, using Lemma 3.6, we see that, for all $v \in \mathbb{R}^n$,

$$0 = [c(v), g] = [c(v), g]' = -2\,c(i(v)\sigma(g)).$$

Thus $i(v)\sigma(g) = 0$ $\forall v \in \mathbb{R}^n$, and hence $\sigma(g)$ is scalar, i.e., g too is scalar.

Next, let $\{e^1, \ldots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then by using Lemma 3.1 we see that

$$(c(e^i)c(e^j))^* = c(e^j)^* c(e^i)^* = (-c(e^j))(-c(e^i)) = c(e^j)c(e^i) = -c(e^i)c(e^j).$$

It then follows that, for all $a \in Cl^2(\mathbb{R}^n)$,

$$a^* = -a,$$

and hence

$$\exp(a)^* = \exp(a^*) = \exp(-a) = \exp(a)^{-1}.$$

This shows that all the elements of $\mathrm{Spin}(n)$ are unitary elements of $Cl(\mathbb{R}^n)$. As g is a (real) scalar, we deduce that $g = \pm 1$. This proves that $\ker \tau = \{\pm 1\}$ and completes the proof. $\square$

Using τ we can lift any representation of $SO(n)$ to a representation of $\mathrm{Spin}(n)$. Conversely, it follows from Proposition 6.4 that a representation π of $\mathrm{Spin}(n)$ in some vector space E descends to a representation of $SO(n)$ if and only if $\pi(-1) = -1_E$.

Next, any representation of $Cl_{\mathbb{C}}(\mathbb{R}^n)$ restricts to a representation of $\mathrm{Spin}(n)$. In particular, the spinor representation gives rise to a representation:

$$\rho : \mathrm{Spin}(n) \longrightarrow \mathrm{End}(\$).$$

As $\rho(-1) = -\mathrm{id}_{\$}$, we see that this representation does <u>not</u> descend to a representation of $SO(n)$.

Furthermore, as pointed out in the proof of Proposition 6.4, the group $\mathrm{Spin}(n)$ is contained in $Cl^+(\mathbb{R}^n)$, so its action preserves both $\$^+$ and $\$^-$. Therefore, we actually have two representations,

$$\rho_\pm : \mathrm{Spin}(n) \longrightarrow \mathrm{End}(\$^\pm).$$

These representations are called <u>half-spinor representations</u>.

7. Spin Structures and the Dirac Operator:

Let M^n be a C^∞ manifold

Def.: The frame bundle of M is the bundle over M,

$$F(M) := \coprod_{x\in M} F_x(M), \quad F_x(M) := \{\text{Invertible linear map } f : \mathbb{R}^n \to T_xM\}.$$
$$\simeq \{\text{bases of } T_xM\}$$

$F(M)$ is a $GL_n(\mathbb{R})$-principal bundle.

Orientation on M = datum of a non-zero section of $\Lambda^n T^*M$.

= reduction of structure group of $F(M)$ to $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R});\ \det A > 0\}$.

Hence, $F(M) \simeq F^+(M) \times_{GL_n^+(\mathbb{R})} GL_n(\mathbb{R})$

$F_x^+(M) = \{\text{orientation-preserving invertible linear maps } f : \mathbb{R}^n \to T_xM\}$.

$\simeq \{\text{oriented bases of } T_xM\}$ $\qquad F^+(M) = GL_n^+(\mathbb{R})$-principal bundle

Riemannian structure on M = datum of a Riemannian metric on TM

= reduction of structure group of $F(M)$ to $O(n)$

$F(M) \simeq O(M) \times_{O(n)} GL_n(\mathbb{R})$

$O(M) = \{\text{orthonormal bases}\}$ $\qquad O(M) = O(n)$-principal bundle

Oriented Riemannian structure on M = reduction of structure group of $F(M)$ to $SO(n) = O(n) \cap GL_n^+(\mathbb{R})$

$F(M) \simeq SO(M) \times_{SO(n)} GL_n(\mathbb{R})$

$SO_x(M) = \{\text{orientation isometric linear map } f : \mathbb{R}^n \to T_xM\}$

Def.: A spin structure on M is a reduction of a structure group of $F(M)$ to $\mathrm{Spin}(n)$, i.e., $\exists\ \mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(M)$ s.t. $F(M) \simeq \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} GL_n(\mathbb{R})$.

If M is spin, then the spinor bundle can be realized as an associated bundle

$$\$ = \$(M) := \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \$(\mathbb{R}^n)$$

spinor space of $\mathbb{R}^n$

Rmk: Spinor representation $\rho : \mathrm{Spin}(n) \to \$(\mathbb{R}^n)$ unitary so get natural Hermitian inner product on $\$$.

Suppose M oriented + Riem.:

Def.: The Clifford bundle $Cl(M)$ is the subbundle of $\mathrm{End}(\Lambda^* T^*M)$ whose fiber at x is the algebra spanned by

$$c(v) := \varepsilon(v) - i(v), \qquad v \in T_x^*M$$

$$\varepsilon(v)\xi := v \wedge \xi \quad \forall \xi \in \Lambda^* T_x^*M$$

$$i(v)(\xi^1 \wedge \dots \wedge \xi^l) = \sum_{j=1}^{l} (-1)^{j-1} \langle v, \xi^j \rangle\, \xi^1 \wedge \dots \wedge \widehat{\xi^j} \wedge \dots \wedge \xi^l \quad \forall \xi^j \in T_x^*M.$$

$Cl(M)$ is a $Cl(\mathbb{R}^n)$-principal bundle and a $SO(M)$-principal bundle.

$$Cl(M) = SO(M) \times_{SO(n)} Cl(\mathbb{R}^n).$$

We have vector bundle isomorphisms,

$$\Lambda T^*M \underset{\sigma}{\overset{c}{\rightleftarrows}} Cl(M). \qquad \Lambda T^*M \cong SO(M)\times_{SO(n)} \Lambda^* \mathbb{R}^n$$

Suppose in addition that M is <u>spin</u>. Then

$$SO(M) \cong Spin(M) \times_{Spin(n)} SO(n)$$

$$\rightsquigarrow \quad \underset{Cl(M)\otimes\mathbb{C}}{\underset{\|}{Cl_{\mathbb{C}}(M)}} \cong Spin(M)\times_{Spin(n)} Cl_{\mathbb{C}}(\mathbb{R}^n) \cong Spin(M)\times_{Spin(n)} End(\$(\mathbb{R}^n)) \cong End(\$).$$

Get $*$-action

$$c : \Lambda T^*M \otimes \$ \longrightarrow \$ \qquad c(\xi)^* = -c(\xi) \quad \forall \xi \in C^\infty(M, T^*M)$$

$$(\xi, \sigma) \longmapsto c(\xi)\sigma.$$

In addition, Levi-Civita connection on $SO(M)$ lifts to a connection on $Spin(M)$ by using

$$\tau^{-1} : so(n) \longrightarrow Cl^2(\mathbb{R}^n) = \text{Lie alg. of } Spin(n)$$

$$\tau^{-1}(A) = \frac{1}{2}\sum_{i<j} \langle Ae^i, e^j\rangle c(e^i)c(e^j).$$

<u>Consequence</u>: The Levi-Civita connection (covariant derivative) ∇^{TM} lifts to a unitary connection (covariant derivative) on $\$$:

$\{e_i\}$ local orthonormal frame of TM $\qquad \omega_{ij\ell} = \langle \nabla^{TM} e_j, e_\ell \rangle \qquad \nabla^{TM} : C^\infty(M, TM) \longrightarrow C^\infty(M, T^*M \otimes TM)$

$\{e^i\}$ dual orth. frame of T^*M $\qquad \nabla^{\$} : C^\infty(M, \$) \longrightarrow C^\infty(M, T^*M \otimes \$)$

$$\nabla^{\$}_{\partial_i} = \partial_i + \frac{1}{4}\sum \omega_{ij\ell}\, \underbrace{c(e^j)c(e^\ell)}_{\in End(\$)}.$$

<u>Def. (Dirac Operator)</u>: The Dirac operator is the first-order differential operator

$$D\!\!\!/ : C^\infty(M, \$) \longrightarrow C^\infty(M, \$),$$

given by the composition

$$D\!\!\!/ : C^\infty(M, \$) \xrightarrow{\nabla^{\$}} C^\infty(M, T^*M \otimes \$) \xrightarrow{c} C^\infty(M, \$).$$

<u>Rmk</u>: Locally,

$$D\!\!\!/ = \sum c(dx^i)\nabla^{\$}_{\partial_i} = \sum c(e^j)\nabla^{\$}_{e_j}$$

<u>Prop 7.1</u>: (1) $D\!\!\!/$ is (formally) selfadjoint.

(2) $\sigma_1(D\!\!\!/)(x, \xi) = i\,c(\xi) \leftarrow$ Clifford multiplication by $\xi \in T^*_x M$

(3) $D\!\!\!/$ is elliptic.

(4) $[D\!\!\!/, f] = c(df) \ \forall f \in C^\infty(M)$.

Proof: (2): $D\!\!\!/ = \sum c(dx^i)\nabla^{\$}_{\partial_i}$ $\quad \nabla^{\$}_{\partial_i} = \partial_i + \text{l.o.t.}$

$$\sigma_1(D\!\!\!/) = \sum c(dx^j)\, i\xi_j = ic\big(\underbrace{\textstyle\sum \xi_j dx^j}\big) = ic(\xi)$$

(3) $\underset{\sigma_1(D\!\!\!/^2)}{\sigma_1(D\!\!\!/)^2} = (ic(\xi))(ic(\xi)) = -\underbrace{c(\xi)^2}_{\langle \xi, \xi\rangle}\xi = |\xi|^2 \Rightarrow \sigma_1(D\!\!\!/)$ invertible

$\Rightarrow D\!\!\!/$ is elliptic

(4) follows from (1) $[D\!\!\!/, f] = \sigma_1(x, df) = ic(df)$

general fact for 1st order diff. op.

Thm 2 (Lichnerowicz Formula): We have
$$D\!\!\!/^2 = (\nabla^{\$})^*\nabla^{\$} + \frac{1}{4}\kappa,$$
where κ is the scalar curvature of M.

Assume M even. Then $\$ = \$^+ \oplus \$^-$ compatible with action of $Cl(M)$

$\rightsquigarrow \$_c$ Clifford module.

$\nabla^{\$}$ preserves splitting $\$ = \$^+ \oplus \$^-$

$c(e^i)$ switches $\$^\pm$ and $\$^\mp$.

Therefore,
$$D\!\!\!/ = \begin{pmatrix} 0 & D\!\!\!/_- \\ D\!\!\!/_+ & 0 \end{pmatrix} \quad \text{w/} \quad D\!\!\!/_\pm : C^\infty(M, \$^\pm) \longrightarrow C^\infty(M, \$^\mp).$$

8. Dirac Operators on Clifford-module bundle:

M^n = Riem. mfld of even dim. $Cl_c(M)$ = complexified Clifford algebra bundle of M.

Def.: A Hermitian vector bundle $E \to M$ is $\mathbb{Z}_2$-graded if it admits an orthogonal splitting
$$E = E^+ \oplus E^-, \quad E^\pm \text{ subbundle of } E.$$

Def.: Let E be a $\mathbb{Z}_2$-graded Hermitian vector bundle. A Clifford bundle structure on E is given by a Clifford action, i.e. algebra-bundle morphism,
$$\pi : Cl_c(M) \longrightarrow \mathrm{End}(E), \quad \pi(a)\xi = a\cdot\xi \quad a \in Cl_c(M),\ \xi \in E$$
such that:

(i) π is a graded action, i.e., $\pi(Cl_c^\pm(M)) \subset \mathrm{End}(E)^\pm$

(ii) c is a $*$-morphism, i.e., $\pi(a^*) = \pi(a)^* \ \forall a \in Cl(M)$ ($\Leftrightarrow \pi(c(\omega))^* = -\pi(c(\omega)) \ \forall \omega \in T^*M$)

Ex1: $E = \Lambda^*_c T^*M = \Lambda^{ev}T^*M \oplus$ with π = inclusion $Cl_c(M) \hookrightarrow \mathrm{End}(\Lambda^*_c T^*M)$.

<u>Ex. 2</u>: (M oriented) $E = \Lambda^{\bullet}_{\mathbb{C}} T^* M$ w/ same action as in Ex. 1, but w/ grading

$\Lambda^{\bullet}_{\mathbb{C}} T^* M = \Lambda^+ T^* M \oplus \Lambda^- T^* M$, $\quad \Lambda^{\pm} T^* M = \ker(\Gamma \mp 1)$,

Γ = chirality op. $= i^{\frac{n}{2}} c(\omega_M)$ $\quad \omega_M$ = volume form $= \sqrt{\det g(x)}\, dx^1 \wedge \cdots \wedge dx^n$

$\Gamma \in C^\infty(M, \mathrm{End}(\Lambda^{\bullet}_{\mathbb{C}} T^* M))$ $\quad = e^1 \wedge \cdots \wedge e^n$ $\{e^i\}$ oriented orth. frame of T^*M

<u>Rmk</u>: $\Gamma = i^{\frac{n}{2}} *$, $\quad *$ = Hodge star operator $\quad * : \Lambda^k T^* M \longrightarrow \Lambda^{n-k} T^* M$

$*\alpha \wedge \beta = \langle \alpha, \beta \rangle \omega_M$. $\quad \alpha, \beta \in \Lambda^k T^*_x M$

<u>Ex. 3</u>: M spin, oriented. W = $\mathbb{Z}_2$-graded oriented Hermitian vector bundle

$E = \$ \otimes W \quad (\$ \otimes W)^{\pm} = (\$^{\pm} \otimes W^+) \oplus (\$^{\mp} \otimes W^-)$

" $a.(\sigma \otimes w) = (\rho(a)\sigma) \otimes w \qquad \rho : Cl_c(M) \longrightarrow \mathrm{End}(\$)$ spinor representation.

$E = \$ \otimes W$ = twisted Clifford-module bundle.

<u>Rmk</u>. Locally, $M \simeq U \subset \mathbb{R}^n$ $\mathbb{R}^n$ is spin so locally spinor bundle $\$$ always exists

<u>Prop</u>. Locally any Clifford-module bundle is of the form $\$ \otimes W$.

<u>Def.</u>: Let E be a Clifford-module bundle. A <u>Clifford connection</u> on E is a connection $\nabla^E : C^\infty(M, E) \longrightarrow C^\infty(M, T^*M \otimes E)$ which preserves its Clifford-module bundle structure, that is, for all $X \in C^\infty(M, TM)$,

(i) $\nabla_X (C^\infty(M, E^{\pm})) \subset C^\infty(M, E^{\pm})$.

(ii) $\langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle = X(\langle \xi, \eta \rangle) \quad \forall \xi, \eta \in C^\infty(M, E)$

(iii) $[\nabla_X, \pi(a)] = \pi(\nabla_X^{Cl(M)} a) \quad \forall a \in C^\infty(M, Cl_c(M))$ $\quad$ ($\nabla^{Cl(M)}$ = lift of LC connection to $Cl(M)$).

<u>Ex. 1</u>: $E = \Lambda^{\bullet} T^* M$, $\nabla^{\Lambda^{\bullet} T^* M}$ = Levi-Civita connection lifted to $\Lambda^{\bullet} T^* M$ so $\nabla^{\Lambda^{\bullet} T^* M}$ is a Clifford connection.

<u>Ex 2</u>: M spin + oriented W = $\mathbb{Z}_2$-graded Herm. ∇^W connection on W satisfying (i) + (ii).

$\nabla^{\$ \otimes W} := \nabla^{\$} \otimes 1 + 1 \otimes \nabla^W$ is a Clifford connection on $\$ \otimes W$.

<u>Prop</u>. Locally, any Clifford connection is of the form $\nabla^{\$ \otimes W}$.

Let E be a Clifford-module bundle w/ Clifford connection ∇^E.

$R^M \in C^\infty(M, \Lambda^2 T^* M \otimes \mathrm{End}(TM))$ = Riemannian curvature

We can lift R^M to E as $R^E \in C^\infty(M, \Lambda^2 T^* M \otimes \mathrm{End}(E))$ $\in \Lambda^2 T^* M$ $\in \mathrm{End}(E)$

$$R^E = \frac{1}{4} \sum_{k,l} \langle R(\partial_i, \partial_j) e_k, e_l \rangle (dx^i \wedge dx^j) \otimes (\pi(c(e^k)) \pi(c(e^l)))$$

<u>Ex</u>: $E = \$$, $\quad R^{\$}$ = curvature of spin connection $\nabla^{\$}$.

Prop. The curvature $F^E \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}(E))$ of F^E is of the form
$$F^E = R^E + F^{E/\$},$$
where $F^{E/\$} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{Hom}_{Cl(M)}(E,E))$ is called the twisting curvature.

Rmk: $\mathrm{Hom}_{Cl(M)}(E,E)_x = \{ T \in \mathrm{End}(E)_x ;\ [T, \pi(a)] = 0 \ \forall a \in Cl_x(M) \}$.

Ex: $E = \$ \otimes W \rightsquigarrow F^{E/\$} = 1 \otimes F^W$, F^W = curvature of ∇^W.
$\nabla^E = \nabla^\$ \otimes 1 + 1 \otimes \nabla^W$

Def. The Dirac operator associated to (E, ∇^E) is the 1st order differential operator $D_E : C^\infty(M,E) \to C^\infty(M,E)$ given by the composition,
$$C^\infty(M,E) \xrightarrow{\nabla^E} C^\infty(M, T^*M \otimes E) \xrightarrow{\pi \circ c} C^\infty(M,E)$$
$$v \otimes \xi \longrightarrow \pi(c(v))\xi$$

Rmk: Locally, $D_E = \sum \pi(c(dx^i)) \nabla_i^E$.

Ex.1: $E = \Lambda^*_{\mathbb{C}} T^*M \rightsquigarrow D_E = d + d^*$ (de Rham operator) d = exterior differential
$d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$.

Ex.2: $E = \$ \otimes W$ $\nabla^E = \nabla^\$ \otimes 1 + 1 \otimes \nabla^W$
Dirac op. $\not{D}_M : C^\infty(M, \$) \xrightarrow{\nabla^\$} C^\infty(M, T^*M \otimes \$) \xrightarrow{c} C^\infty(M, \$)$
$\omega \otimes \sigma \longrightarrow \rho(c(\omega))\sigma$
$D_E = \not{D}_W : C^\infty(M, \$ \otimes W) \xrightarrow{\nabla^\$ \otimes 1 + 1 \otimes \nabla^W} C^\infty(M, T^*M \otimes \$ \otimes W) \xrightarrow{c \otimes 1} C^\infty(M, \$ \otimes W)$
$\omega \otimes \sigma \otimes w \longrightarrow (\rho(c(\omega))\sigma) \otimes w$

$\rightsquigarrow \not{D}_W = \not{D}_M \otimes 1 + (c \otimes 1)(1 \otimes \nabla^W)$ = twisted Dirac op.

Rmk: Locally, D_E is of the $\not{D}_W$.

Prop. (1) D_E is selfadjoint and elliptic.
(2) $D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}$, $D_E^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$, $D_E^- = (D_E^+)^*$

Prop. (Lichnerowicz Formula):
$$(D_E)^2 = (\nabla^E)^* \nabla^E + \frac{1}{4}\kappa + \mathcal{F}^{E/\$},$$
$\mathcal{F}^{E/\$} \in C^\infty(M, \mathrm{End}(E))$
$\mathcal{F}^{E/\$} = \frac{1}{2} \sum_{i<j} F^{E/\$}(e_i, e_j)\pi(c(e^i)c(e^j))$ $\{e_i\}$ loc. orth. frame of TM
$\{e^i\}$ dual coframe of T^*M

9. The Index of a Fredholm Operator:

$\mathcal{H}_1, \mathcal{H}_2 =$ Hilbert spaces

Def.: (1) An operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm when

$$\dim \ker T < \infty \quad \text{and} \quad \dim \underbrace{\operatorname{coker} T}_{= \mathcal{H}_2 / \operatorname{im} T} < \infty$$

(2) The set of Fredholm op. $T: \mathcal{H}_1 \to \mathcal{H}_2$ is denoted $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$.

(3) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm, it index is

$$\operatorname{ind} T := \dim(\ker T) - \dim(\operatorname{coker} T).$$

Rmk: $T \in \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2) \Longrightarrow \operatorname{im} T$ is closed and $\ker T^* \simeq \operatorname{coker} T$. Thus

$$T \in \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2) \Longrightarrow \boxed{\operatorname{ind} T = \dim \ker T - \dim \ker T^*}$$

Prop.: (1) $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ is an open subset of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$

(2) $\operatorname{ind}: \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ is continuous and constant on path-connected components of $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$.

(3) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$ and $S \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_2)$ are Fredholm, then $ST \in \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ and

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$$

(4) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is invertible modulo compact operators, then T is Fredholm and

$$\operatorname{ind}(T+R) = \operatorname{ind}(T) \quad \forall R \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

$M =$ compact (Riemann) manifold

$E_1, E_2 =$ (Hermitian) vector bundles over M

Prop.: Let $P: C^\infty(M, E_1) \to C^\infty(M, E_2)$ be an elliptic operator of order m, $m > 0$ and let $P^*: C^\infty(M, E_2) \to C^\infty(M, E_1)$ be its formal adjoint.

(1) $\forall s \in \mathbb{R}$ the operator $P: L^2_{s+m}(M, E_1) \to L^2_s(M, E_2)$ is Fredholm and its index does not depend on s.

(2) $\ker P$ and $\ker P^*$ are contained in $C^\infty(M, E_1)$ and $C^\infty(M, E_2)$ respectively and

$$\operatorname{ind} P = \dim \ker P - \dim \ker P^*$$

(3) $\operatorname{ind}(P+R) = \operatorname{ind} P \quad \forall R \in \Psi^{m-1}(M, E_1, E_2)$.

Rmk: (3) shows that $\operatorname{ind} P$ is an invariant of the principal symbol $\sigma_m(P)$ of P. This is an a homotopy invariant of $\sigma_m(P)$.

Question (Gel'fand): Compute the index of a general elliptic ΨDO.

- Solution provided by the Atiyah-Singer index theorem (ASIT).
- Geometric meaning of ASIT is reached by the Local Index Formula (LIF). The LIF is actually equivalent to the ASIT.
- The fact the LIF is equivalent to the ASIT.

CHAPTER 8

The Atiyah-Singer Index Theorem

The main reference for this chapter is the monograph [**BGV**]. We also refer to [**Sh**, §8] for the section on Fredholm operators. The proof of the Atiyah-Singer's local index formula given in Section 8.12 is taken from [**Po**].

We refer to the scans on the course's website for the notes of Sections 1–9 about Clifford algebras, spin structures, Dirac operators and Fredholm operators, and for the notes of the appendix on characteristic classes.

8.10. The Local Index Formula of Atiyah-Singer

Let (M^n, g) be an even dimensional compact oriented Riemannian manifold and let $E = E^+ \oplus E^-$ be a Clifford-module bundle equipped with a Clifford connection ∇^E. We shall denote by $D_E : C^\infty(M, E) \rightarrow C^\infty(M, E)$ the associated Dirac operator as defined in Section 8.

As we saw in Section 8, the operator D_E is elliptic and takes the form,

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}, \qquad D_E^\pm : C^\infty(M, E^\pm) \rightarrow C^\infty(M, E^\pm).$$

Moreover, the selfadjointness of D_E implies that $(D_E^\pm)^* = D_E^\mp$.

As D_E is elliptic, the results of Section 9 show that D_E is Fredholm. However, as D_E is selfadjoint its index must be zero. However, the ellipticity of D_E implies that $D_E^\pm$ is elliptic, and hence is Fredholm. We then define the index of D_E to be

$$\operatorname{ind} D_E := \operatorname{ind} D_E^+ = \dim\ker D_E^+ - \dim\ker D_E^-,$$

where we have used the fact that D_E^- is the adjoint of D_E^+.

Let us denote by $\operatorname{Str}_{E/\$}$ the relative supertrace on the fibers of $\operatorname{Hom}_{\mathrm{Cl}_{\mathbb{C}}(M)}(E, E)$. Recall that

$$\operatorname{Str}_{E/\$}(T) = 2^{-\frac{n}{2}} \operatorname{Str}_E(\Gamma T) \qquad \forall T \in \mathbb{C}^\infty(M, \operatorname{Hom}_{\mathrm{Cl}_{\mathbb{C}}(M)}(E, E)),$$

where we have denoted by Γ the chirality operator (or, more precisely, the section $\operatorname{End} E$ given by the pointwise action of chirality operator on the fibers of E).

In the sequel, we denote by $F^{E/\$}$ the twisted curvature of the Clifford connection ∇^E (see Section 8 for its precise definition).

Definition 8.10.1. *The relative Chern form of the twisted curvature $F^{E/\$}$ is*

$$\operatorname{Ch}(F^{E/\$}) = \operatorname{Str}_{E/\$}\left[e^{-F^{E/\$}}\right] \in C^\infty(M, \Lambda^{ev} T^* M).$$

For instance, suppose that M is spin and let $W = W^+ \otimes W^-$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle with Hermitian connection ∇^W preserving its $\mathbb{Z}_2$-grading.

Then we can we form the twisted Clifford-bundle module $E = \not{S} \otimes W$ and equipped with the twisted Clifford connection,

$$\nabla^E = \nabla^{\not{S}} \otimes 1_W + 1_{\not{S}} \otimes \nabla^W.$$

As alluded to in Section 8, the twisted curvature of this twisted connection is just $1_{\not{S}} \otimes F^W$. Therefore, its relative Chern form is equal to

$$\begin{aligned} \mathrm{Ch}(F^{E/\not{S}}) &= \mathrm{Str}_{E/\not{S}}\left[1_{\not{S}} \otimes e^{-F^W}\right] = \mathrm{Str}_W\left[e^{-F^W}\right] \\ &= \mathrm{Tr}_{W^+}\left[e^{-F^{W^+}}\right] - \mathrm{Tr}_{W^-}\left[e^{-F^{W^-}}\right] \\ &= \mathrm{Ch}(F^{W^+}) - \mathrm{Ch}(F^{W^-}), \end{aligned}$$

where $F^{W^\pm}$ is the curvature of the connection on $W^\pm$ induced by ∇^W and $\mathrm{Ch}(F^{W^\pm})$ is its usual Chern form.

In addition, as explained in the appendix on characteristic classes, the $\hat{A}$-form of the Riemann curvature R^M of M (i.e., the curvature of the Levi-Civita connection on TM) is

$$\hat{A}(R^M) := \det{}^{\frac{1}{2}}\left[\frac{R^M/2}{\sinh(R^M/2)}\right].$$

We are now in a position to state the index formula of Atiyah-Singer.

Theorem 8.10.2 (Atiyah-Singer). *We have*

$$\mathrm{ind}\, D_E = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}, \tag{8.1}$$

where $\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ *denotes the n-th degree component of the even form* $\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}})$.

Remark 8.10.3. The local index formula of Atiyah- Singer continue to hold even M is non-orientable. Notice that the integrand $\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ defines a density, namely,

$$\langle \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}, \omega_M \rangle v_g(x), \tag{8.2}$$

where ω_M is the volume form and $v_g(x)$ is the Riemannian density. The integral $\int_M \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ then is the integral of this density.

The above process allows us to identify n-the degree forms and densities, but it depends on a choice of orientation. Observe that the definition of the relative Chern form $\mathrm{Ch}(F^{E/\not{S}}$ depends on the definition of relative supertrace which involves the chirality operator, hence depends on the choice of the orientation (see Eq. (8.6). However, if at a point $x \in M$, we reverse the orientation of T_xM then it only affects by a change of sign the values at x of the chirality operator Γ and relative Chern form $\mathrm{Ch}(F^{E/\not{S}}$. Since this similarly affects the value at x of the volume form, we see that the value at x of the density (8.2) is actually independent of the orientation of T_xM. Therefore (8.2) defines a density even M is non-orientable. Then the index of D_E continue to be given by the formula (8.1), where the r.h.s. is interpreted as a multiple of the integral of the density (8.2).

The above index formula is often referred to as the *local index formula* of Atiyah-Singer. It is not a mere special case of the *full index theorem* of Atiyah-Singer [**AS**] for general elliptic ΨDOs. In many respects, the local index formula is equally important, if not even more important, than the full index theorem.

As it turns out, K-theoretic arguments show that the local index formula is actually equivalent to the full index theorem (see, e.g., [**ABP**]). More importantly, its only the case of Dirac operators on Clifford-module bundles that the Atiyah-Singer's index theorem reaches its true geometric contents. Therefore, the local index formula for Dirac operators is often confused with the full index theorem of Atiyah-Singer.

We shall describe some of the geometric applications of the Atiyah-Singer's local index formula in the next section. Before doing this let us briefly outline of the proof this formula.

As D_E^2 is a selfadjoint operator with non-negative spectrum, the Borel functional calculus allows us to define the heat semi-group $e^{-tD_E^2}$, $t \geq 0$, as a family of bounded operators on $L^2(M,E)$ (cf. Chapter II). Indeed, if $(\xi_k)_{k\geq 0}$ is an orthonormal basis of eigenvectors such that $D_E^2\xi_k = \lambda_k(D_E^2)\xi_k$ for all $k \geq 0$, then

$$e^{-tD_E^2}\xi_k = e^{-t\lambda_k(D_E^2)}\xi_k \qquad \forall k \geq 0.$$

In addition,we denote by Str the supertrace on the $\mathbb{Z}_2$-graded Hilbert space $L^2(M,E) = L^2(M,E^+) \oplus L^2(M,E^-)$, that is,

$$\operatorname{Str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{Trace}(A) - \operatorname{Tr}(B)$$

for any trace-class operator $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $L^2(M,E) = L^2(M,E^+) \oplus L^2(M,E^-)$.

Let $t > 0$. The ellipticity of D_E^2 ensures us that, for $t > 0$, the operator $e^{-tD_E^2}$ is smoothing, and hence has a smooth Schwartz kernel, i.e., a section $k_t(x,y)$ in $C^\infty(M \times M, E \boxtimes (E^* \otimes |\Lambda|(M)))$, which is called the *heat kernel* of D_E^2 (see Section 8.12 on this point). It then follows that $e^{-tD_E^2}$ is trace-class and its supertrace is given by

$$\operatorname{Str} e^{-tD_E^2} = \int_M \operatorname{Str}_E[k_t(x,x)],$$

where $\operatorname{Str}_E[k_t(x,x)]$ is defined as a density on M.

THEOREM 8.10.4 (McKean-Singer Formula). *For all $t > 0$,*

$$\operatorname{ind} D_E = \operatorname{Str} e^{-tD_E^2} = \int_M \operatorname{Str}_E[k_t(x,x)].$$

PROOF. We know that the Dirac operator D_E is of the form,

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix},$$

where $D_E^\pm : C^\infty(M,E^\pm) \to C^\infty(M,E^\pm)$ and $(D_E^\pm)^* = D_E^\mp$. Thus,

$$D_E^2 = \begin{pmatrix} D_E^- D_E^+ & 0 \\ 0 & D_E^+ D_E^- \end{pmatrix}.$$

Let $t > 0$. Then

$$e^{-tD_E^2} = \begin{pmatrix} e^{-tD_E^- D_E^+} & 0 \\ 0 & e^{-tD_E^+ D_E^-} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{Str} e^{-tD_E^2} &= \operatorname{Trace}\left[e^{-tD_E^- D_E^+}\right] - \operatorname{Trace}\left[e^{-tD_E^+ D_E^-}\right] \\ &= \sum_{\lambda \in \mathbb{C}} e^{-t\lambda} \dim E_\lambda^+ - \sum_{\lambda \in \mathbb{C}} e^{-t\lambda} \dim E_\lambda^- \\ &= \dim E_0^+ - \dim E_0^- + \sum_{\lambda \in \mathbb{C} \setminus 0} e^{-t\lambda} (\dim E_\lambda^+ - \dim E_\lambda^-), \end{aligned}$$

where, for all $\lambda \in \mathbb{C}$, we have set

$$E_\lambda^+ := \ker(D_E^- D_E^+ - \lambda) \quad \text{and} \quad E_\lambda^- := \ker(D_E^+ D_E^- - \lambda).$$

Notice that, as $(D_E^\pm)^* = D_E^\mp$, we have

$$E_0^\pm = \ker D_E^\mp D_E^\pm = \ker (D_E^\pm)^* D_E^\pm = \ker D_E^\pm.$$

Thus,

$$\dim E_0^+ - \dim E_0^- = \dim \ker D_E^+ - \dim \ker D_E^- = \operatorname{ind} D_E,$$

and hence

$$\operatorname{Str} e^{-tD_E^2} = \operatorname{ind} D_E + \sum_{\lambda \in \mathbb{C} \setminus 0} e^{-t\lambda} (\dim E_\lambda^+ - \dim E_\lambda^-). \tag{8.3}$$

Next, let $\lambda \in \mathbb{C} \setminus 0$ and let ξ be in $E_\lambda^+ = \ker(D_E^- D_E^+ - \lambda)$. Then

$$D_E^+ D_E^- (D_E^+ \xi) = D_E^+ (D_E^- D_E^+ \xi) = \lambda D_E^+ \xi,$$

that is, $D_E^+ \xi$ is contained in $E_\lambda^- = \ker(D_E^+ D_E^- - \lambda)$. Thus D_E^+ induces a linear map from E_λ^+ to E_λ^-.

Similarly, the operator D_E^- induces a linear map from E_λ^- and E_λ^+. As by the very definitions of E_λ^+ and E_λ^- we have

$$\left(D_E^- D_E^+\right)_{|E_\lambda^+} = \lambda \operatorname{id}_{E^+} \quad \text{and} \quad \left(D_E^+ D_E^-\right)_{|E_\lambda^-} = \lambda \operatorname{id}_{E^-},$$

we see that D_E^+ induces a linear isomorphism from E_λ^+ onto E_λ^-. Thus,

$$\dim E_\lambda^+ = \dim E_\lambda^- \qquad \forall \lambda \in \mathbb{C} \setminus 0.$$

Combining this with (8.3) yields the the McKean-Singer formula. □

Now, the local index formula of Atiyah-Singer is proved by combining the McKean-Singer formula with the following:

THEOREM 8.10.5 (Patodi, Gilkey, Atiyah-Bott-Patodi). *In $C^\infty(M, |\Lambda|(M))$,*

$$\operatorname{Str}_E[k_t(x,y)] \longrightarrow \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^{E/\not{S}})\right]^{(n)} \qquad \textit{as } t \to 0^+.$$

Theorem 8.10.5 is often referred to as the *local index theorem*. In the above general form, it was first proved by Atiyah-Bott-Patodi [**ABP**] by making use of Riemannian invariant theory (see also [**Gi**]). The first purely analytical proofs were produced by Getzler [**Ge1**] and Bismut [**Bi**] by means of different approaches. These proofs were further simplified by the celebrated"short proof" of Getzler [**Ge2**], where was introduced the use of the so-called "Getzler's rescaling". This proof paved the way for many generalizations of the local index formula of Atiyah-Singer'.

In Section 8.12, we reproduce the proof of the local index theorem given in [**Po**]. This proof is especially relevant for applications in noncommutative geometry (cf. Chapter 11).

Notice, that it can be shown that in $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$,

$$k_t(x,x) \sim t^{-\frac{n}{2}} \sum_{j\geq 0} t^j a_j(D_E^2)(x) \qquad \text{as } t \to 0^+,$$

where $a_0(D_E^2)(x)$ is the positive multiple of the Riemannian density (cf. Section 8.12). Therefore, it is a very striking fact that under the supertrace all the divergent terms in the above asymptotics vanish and we obtain a convergent quantity.

It should also be stressed out that this phenomenon is really specific to Dirac operators coming from Clifford connections; in general the local index theorem does not occur for other operators.

8.11. Geometric Applications

Let (M^n, g) be an even-dimensional oriented compact Riemannian manifold. Depending on the existence of an additional structure (e.g.spin structure, complex structure) the Atiyah-Singer index formula has various striking consequences. We give here a brief overview of them, referring to the book of Berline-Getzler-Vergne for a more detailed treatment, including proofs of the results stated here.

8.11.1. Dirac Operators on Spin Manifolds. Assume that M is *spin* and let us denote by $\not{S}$ its spinor bundle. In addition, let $W = W^+ \otimes W^-$ be a Hermitian $\mathbb{Z}_2$-graded vector bundle equipped with a Hermitian connection ∇^W preserving the $\mathbb{Z}_2$-grading of W.

We denote by $\mathcal{D}_W : C^\infty(M, \not{S} \otimes W) \to C^\infty(M, \not{S} \otimes W)$ the associated twisted Dirac operator, that is, $\mathcal{D}_W$ is the Dirac operator associated to the twisted Clifford connection $\nabla^{\not{S}\otimes W} := \nabla^{\not{S}} \otimes 1 + 1 \otimes \nabla^W$, where $\nabla^{\not{S}}$ is the spin connection.

As shown before, the twisted curvature of $\nabla^{\not{S}\otimes W}$ is $1 \otimes F^W$, where F^W is the curvature of the connection ∇^W, and its relative Chern form agrees with the Chern form $\operatorname{Ch}(F^W)$ of F^W. Therefore, we the local index formula of Atiyah-Singer gives

THEOREM 8.11.1 (Atiyah-Singer). *We have*

$$\operatorname{ind} \not{D}_W = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^W)\right]^{(n)}.$$

In particular, in the case of the Dirac operator $\not{D}_M$ of M acting on spinors (i.e., when W is the flat trivial line bundle over M) we get

THEOREM 8.11.2 (Atiyah-Singer). *We have*

$$\operatorname{ind} \not{D}_M = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M)^{(n)}.$$

As an immediate corollary we obtain:

COROLLARY 8.11.3. *If the $\hat{A}$-number of M, i.e.,*

$$\hat{A}(M) := \int_M \left[\hat{A}(R^M)\right]^{(n)},$$

is not an integer, then M does not admit a spin structure.

In the sequel, we denote by κ_M the scalar curvature of M.

THEOREM 8.11.4 (Lichnerowicz). *Suppose that $\kappa_M(x) > 0$ for all $x \in M$. Then the null space of $\not{D}_M$ is reduced to $\{0\}$.*

PROOF. This is a direct consequence of the Lichnerowicz's formula,

$$\not{D}_M^2 = (\nabla^{\not{S}})^* \nabla^{\not{S}} + \frac{1}{4}\kappa_M.$$

It implies that, for all $u \in C^\infty(M, \not{S})$,

$$\langle \not{D}_M u, u \rangle = \langle \nabla^{\not{S}} u, \nabla^{\not{S}} u \rangle + \frac{1}{4}\langle \kappa_M u, u \rangle \geq \frac{1}{4} \int \kappa_M(x) \| u(x) \|^2_{\not{S}_x} v_g(x) \geq c \| u \|^2_{L^2(M, \not{S})},$$

where $v_g(x)$ is the Riemannian density and we have set $c := \inf_{x \in M} \kappa_M(x)$.

As M is compact and κ_M is positive, the constant c is positive. Therefore, no non-zero smooth section of $\not{S}$ can be contained in $\ker \not{D}_M^2$. As $\not{D}_M$ is elliptic and selfadjoint, $\ker \not{D}_M^2 = \ker \not{D}_M \subset C^\infty(M, \not{S})$, so we see that $\ker \not{D}_M$ is trivial. $\square$

If $\not{D}_M$ has a trivial null space, then, obviously, $\operatorname{ind} \not{D}_M = 0$. Therefore, combining Theorem 8.11.2 and Theorem 8.11.4 gives

THEOREM 8.11.5. *If M carries a metric of positive scalar curvature, then its $\hat{A}$-number $\hat{A}(M)$ vanishes.*

8.11.2. The Chern-Gauss-Bonnet Theorem. The (complexified) exterior-algebra bundle $\Lambda^*_{\mathbb{C}} T^* M$ carries the $\mathbb{Z}_2$-grading,

$$\Lambda^* T^* - C^\infty(M) = \Lambda^{\mathrm{ev}} T^* M \oplus \Lambda^{\mathrm{odd}} T^* M.$$

This is a Clifford-module bundle with respect to the natural action of $\mathrm{Cl}_{\mathbb{C}}(M)$ seen as a sub-bundle of $\mathrm{End}(\Lambda^*_{\mathbb{C}} T^* M)$. Because we can define another $\mathbb{Z}_2$-grading by means of the chirality operator (see next subsection), we shall denote by $\Lambda^{\mathrm{ev/odd}} T^*_{\mathbb{C}} M$, or simply$\Lambda^{\mathrm{ev}/odd}$ the Clifford-module bundle defined by means of the above $\mathbb{Z}_2$-grading.

As it turns out, the Levi-Civita connection $\nabla^{\Lambda^*_{\mathbb{C}} T^* M}$ on $\Lambda^{\mathrm{ev}/odd}_{\mathbb{C}} T^* M$ is a Clifford connection and the associated Dirac operator agrees with the de Rham operator,

$$d + d^* : C^\infty(M, \Lambda^* T^* M) \longrightarrow C^\infty(M, \Lambda^* T^* M),$$

where d is the usual exterior differential and d^* is its adjoint (see [**BGV**]).

The index of $d + d^*$ has an important topological interpretation as follows. For $j = 0, \ldots, n$ let us denote by $H^k(M)$ the $(k+1)$-th de Rham cohomology group of M, that is,

$$H^k(M) = \ker d_k / \operatorname{im} d_{k-1},$$

where $d_k : C^\infty(M, \Lambda^k T^* M) \to C^\infty(M, \Lambda^{k+1} T^* M)$ is the de Rham differential on forms of degree k. The *Euler characteristic* of M then is

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k(M).$$

In addition, let $\Delta := (d + d^*)^2 = dd^* + d^* d$ be the Laplace-Beltrami operator and let us denote by $\Delta_k := d_{k-1} d^*_{k+1} + d^*_k d_k$ its restriction to forms of degree k.

PROPOSITION 8.11.6 (Hodge; see [**BGV**]). *We have an orthogonal splitting,*

$$\ker d_k = \operatorname{im} d_{k-1} \oplus \ker \Delta_k,$$

and hence

$$H^k(M) \simeq \ker \Delta_k.$$

That is, any class in $H^k(M)$ can be represented by a unique harmornic form of degree k.

It follows from this that $\dim H^k(M) = \dim \ker \Delta_k$, and hence

$$\begin{aligned}
\chi(M) &= \sum_{k=0}^{n} (-1)^k \dim \ker \Delta_k \\
&= \dim \ker \Delta_{|\Lambda^{\mathrm{ev}} T^*M} - \dim \ker \Delta_{|\Lambda^{\mathrm{odd}} T^*M} \\
&= \dim \ker (d+d^*)_{|\Lambda^{\mathrm{ev}} T^*M} - \dim \ker (d+d^*)_{|\Lambda^{\mathrm{odd}} T^*M} \\
&= \operatorname{ind}(d+d^*).
\end{aligned} \tag{8.4}$$

In the sequel, we denote by $\mathrm{Eul}(R^M)$ the *Euler form* of the curvature R^M of M. Regarding R^M as a section of $(\Lambda^2 T^*M) \otimes \mathrm{End}(TM)$, the *Euler form* is the form of degree of n on M such that, for any local orthonormal frame $\{e_i\}$ of $TM$with dual coframe $\{e^i\}$, we have

$$\frac{1}{(n/2)!} \left(-\frac{1}{4} \sum_{i,j,k,l} R^M_{ijkl} (dx^i \wedge dx^j) \otimes (e^k \wedge e^l) \right)^{\frac{n}{2}} = \mathrm{Eul}(R^M) \otimes (e^1 \wedge \ldots \wedge e^n),$$

where we have set $R^M_{ijkl} := \langle R^M(\partial_i, \partial_j) e_k, e_l \rangle$.

LEMMA 8.11.7 (See [**BGV**]).

(1) *The twisted curvature $F^{(\Lambda^{ev/odd})/\$}$ of the Levi-Civita connection on $\Lambda^{ev/odd}_{\mathbb{C}} T^*M$ is such that, for any local orthonormal frame $\{e_i\}$ of TM with dual coframe $\{e^i\}$, we have*

$$F^{(\Lambda^{ev/odd})/\$} = -\frac{1}{4} \sum_{i,j,k,l} R^M_{ijkl} (dx^i \wedge dx^j) \otimes \left(\varepsilon(e^k) + \iota(e^k) \right) \left(\varepsilon(e^l) + \iota(e^l) \right).$$

(2) *The relative Chern form of the twisted curvature $F^{(\Lambda^{ev/odd})/\$}$ is given by*

$$\mathrm{Ch}\left(F^{(\Lambda^{ev/odd})/\$} \right) = i^{\frac{n}{2}} \, \mathrm{Eul}(R^M). \tag{8.5}$$

As the zeroth-degree component of the even form $\hat{A}(R^M)$ is equal to 1, from (8.5) we get

$$\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\$}) \right]^{(n)} = \left[\hat{A}(R^M) \right]^{(0)} \mathrm{Eul}(R^M) = \mathrm{Eul}(R^M).$$

Therefore, by using (8.4) and applying the local index formula of Atiyah-Singer we obtain

THEOREM 8.11.8 (Chern-Gauss-Bonnet Theorem). *We have*

$$\chi(M) = \operatorname{ind}(d+d^*) = (2\pi)^{-n} \int_M \mathrm{Eul}(R^M).$$

In the above form, the Chern-Gauss-Bonnet theorem is originally due to S.S. Chern. This is the most classical index theorem. When $n = 2$ we recover the well-known Gauss-Bonnet theorem,

$$\chi(M) = \frac{1}{2\pi} \int_M \kappa_M(x) v_g(x),$$

where $v_g(x)$ is the Riemannian density.

8.11.3. Hirzebruch Signature Theorem. Recall that the chirality operator is defined by

$$\Gamma := i^{\frac{n}{2}} c(\omega_M) \in C^\infty(M, \mathrm{Cl}_{\mathbb{C}}(M)) \subset C^\infty(M, \mathrm{End}(\Lambda_{\mathbb{C}} T^*M)), \tag{8.6}$$

where $\omega_M = \sqrt{\det g(x)} dx^1 \wedge \cdots \wedge dx^n$ is the volume form of M. As $\Gamma^2 = 1$ it defines an alternative $\mathbb{Z}_2$-grading on $\Lambda^*_{\mathbb{C}} T^*M$ given by

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^+ T^*_{\mathbb{C}} M \otimes \Lambda^- T^*_{\mathbb{C}} M, \qquad \Lambda^\pm T^*_{\mathbb{C}} M := \ker(\Gamma \mp 1). \tag{8.7}$$

This $\mathbb{Z}_2$-grading is preserved by the action of $\mathrm{Cl}_{\mathbb{C}}(M)$, and so we get a new Clifford-module bundle structure on $\Lambda^*_{\mathbb{C}} T^*M$. We shall denote by $\Lambda^{+/-}_{\mathbb{C}} T^*M$ the $\Lambda^*_{\mathbb{C}} T^*M$ equipped with this Clifford-module bundle structure.

The Levi-Civita connection on $\Lambda^{+/-}_{\mathbb{C}} T^*M$ is a Clifford connection (cf. [**BGV**]), and so the associated Dirac operator is again the de Rham operator $d + d^*$. Notice that, as we are using a different $\mathbb{Z}_2$-grading, the index differs from that in (8.4). As we shall now explain the index that we get is intimately related to the signature of the manifold.

Let $\star \in C^\infty(M, \mathrm{End}\, \Lambda^* T^*M)$ be the Hodge operator, i.e.,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_M \qquad \forall \alpha, \beta \in C^\infty(M, \Lambda^* T^*M),$$

Lemma 8.11.9 (See [**BGV**]). *We have*

$$\Gamma = (-1)^{\frac{k(k-1)}{2}} i^{\frac{n}{2}} \star \qquad \text{on } \Lambda^k_{\mathbb{C}} T^*M,$$
$$d^* = -\star d\star$$

It follows from this lemma that $d + d^* = d - \star d \star$. In the sequel, in order to distinguish with the index in (8.4), we shall denote by $\mathrm{ind}(d - \star d\star)$ the index of $d + d^*$ when using the $\mathbb{Z}_2$-grading defined by the chirality operator. That is,

$$\begin{aligned}\mathrm{ind}(d - \star d\star) &:= \dim \ker(d + d^*)_{|\Lambda^+ T^*_{\mathbb{C}} M} - \dim \ker(d + d^*)_{|\Lambda^- T^*_{\mathbb{C}} M} \\ &= \dim \ker(d - \star d\star)_{|\Lambda^+ T^*_{\mathbb{C}} M} - \dim \ker(d - \star d\star)_{|\Lambda^- T^*_{\mathbb{C}} M}.\end{aligned}$$

It also follows from Lemma 8.11.9 that the Hodge $\star$-operator descends to a linear map,

$$\star : H^k(M) \longrightarrow H^{n-k}(M), \tag{8.8}$$

which is an isomorphism since $\star^2 = (-1)^k$ on $\Lambda^k T^*M$. Using this isomorphism we can prove the nondegeneracy of the bilinear pairing,

$$H^k(M) \times H^{n-k}(M) \ni (\alpha, \beta) \longrightarrow \int_M \alpha \wedge \beta.$$

When n is divisible by 4 this pairing is symmetric on $H^{\frac{n}{2}}(M) \times H^{\frac{n}{2}}(M)$, and hence it gives rise to a nondegenerate quadratic form on $H^{\frac{n}{2}}(M)$. The signature of this quadratic form is called the *signature* of M and is denoted $\sigma(M)$. This is a topological invariant of M.

Lemma 8.11.10 (See [**BGV**]). *If n is divisible by 4, then*

$$\sigma(M) = \operatorname{ind}(d - \star d \star).$$

Because of this lemma, the operator $d - \star d\star$ is often called *signature operator.*

Next, the twisted curvature of the Levi-Civita on $\Lambda^{+/-}T^*_{\mathbb{C}}M$ is continue to be given by (1). As we shall now see, since we are using a different $\mathbb{Z}_2$-grading, we obtain a different relative Chern form, which we shall denote by $\operatorname{Ch}(F^{(\Lambda^{+/-})/\$})$.

Recall that L-form $L(R^M)$ is the of the curvature R^M is defined by

$$L(R^M) := \det{}^{\frac{1}{2}}\left(\frac{R^M/2}{\tanh(R^M/2)}\right).$$

Lemma 8.11.11 (See [**BGV**]). *We have*

$$\operatorname{Ch}(F^{E/\$}) = 2^{\frac{n}{2}} \det{}^{\frac{1}{2}}\left(\frac{\sinh(R^M/2)}{R^M/2}\right) \wedge L(R^M).$$

Observe that $\hat{A}(R^M) \wedge \det^{\frac{1}{2}}\left(\frac{\sinh(R^M/2)}{R^M/2}\right) = 1$. Thus,

$$\hat{A}(R^M) \wedge \operatorname{Ch}(F^{E/\$}) = 2^{\frac{n}{2}} L(R^M).$$

Therefore, using the Atiyah-Singer index formula we obtain:

Theorem 8.11.12 (Atiyah-Singer). *We have*

$$\operatorname{ind}(d - \star d\star) = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M)\right]^{(n)}.$$

Combining this with Lemma 8.11.10, we immediately get

Theorem 8.11.13 (Hirzebruch Signature Theorem). *If $\dim M$ is divisible by 4, then*

$$\sigma(M) = \operatorname{ind}(d - \star d\star) = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M)\right]^{(n)}.$$

Since $\left[L(R^M)\right]^{(4)} = -\frac{1}{96\pi^2}\operatorname{Tr}[(R^M)^2]$, we see that in dimension 4,

$$\sigma(M) = -\frac{1}{24\pi^2}\int_M \operatorname{Tr}[(R^M)^2].$$

More generally, let W be a Hermitian vector bundle equipped with a Hermitian connection ∇^W. Then we can form the twisted Clifford-module bundle $\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W$, where the fiberwise action of $\operatorname{Cl}_{\mathbb{C}}(M)$ is such that, above all $x \in M$,

$$a(\xi \otimes w) = (a\xi) \otimes w, \qquad \forall a \in \operatorname{Cl}_x(M) \quad \forall (\xi, w) \in \Lambda^* T^*_x M \times W_x.$$

The twisted connection $\nabla^{\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W} = \nabla^{\Lambda^* T^* M} \otimes 1 + 1 \otimes \nabla^W$ is a Clifford connection on $\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W$. We shall denote by $(d - \star d\star)_W$ the associated Dirac operator. Its index is denoted $\sigma(M, W)$ and is called the twisted signature with coefficients in W. It depends only on the topology of the manifold M and the vector bundle W.

The twisting curvature $F^{(\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W)/\$}$ is equal to

$$F^{(\Lambda^{+/-})/\$} \otimes 1 + 1 \otimes F^W,$$

where F^W is the curvature of ∇^W. Thus, its relative Chern form is given by

$$\operatorname{Ch}(F^{(\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W)/\$}) = \operatorname{Ch}(F^{E/\$}) \wedge \operatorname{Ch}(F^W).$$

Therefore, we obtain:

THEOREM 8.11.14 (Twisted Signature Theorem). *We have*

$$\sigma(M, W) = (d - \star d\star)_W = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M) \wedge \mathrm{Ch}(F^W)\right]^{(n)}.$$

The twisted signature theorem was originally proved by Atiyah-Singer. It is important because it can be used to prove the *full* Atiyah-Singer index theorem, i.e., the index theorem for *general* elliptic ΨDOs on compact manifolds.

8.11.4. The Hirzebruch-Riemann-Roch Formula. In this subsection, we assume that M is a *complex* manifold of complex dimension n. This means that the tangent bundle TM is thus endowed with a complex structure $J \in C^\infty(M, \mathrm{End}_{\mathbb{R}} TM)$, $J^2 = -1$, so that the holomorphic tangent bundle $T_{1,0}M := \ker(J - i)$ is integrable in Fróbenius' sense, i.e., $[Z, W] \in C^\infty(M, T_{1,0}M)$ for all $Z, W \in C^\infty(M, T_{1,0}M)$.

In addition, we assume that the orientation of M is the orientation defined by its complex structure and the Riemannian metric on $g\ TM$ is the real part of a Hermitian metric h on $T_{\mathbb{C}}M$ with respect to which J is unitary.

The complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ admits the orthogonal decomposition,

$$T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M, \qquad T_{0,1}M := \overline{T_{1,0}M}.$$

By duality this gives rise to orthogonal splittings,

$$T^*_{\mathbb{C}}M = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M, \qquad \Lambda^*_{\mathbb{C}}T^*M = \bigoplus_{q=0}^{n} \Lambda^{p,q}T^*M,$$

where $\Lambda^{p,q}T^*M := \left(\Lambda^{1,0}T^*M\right)^p \wedge \left(\Lambda^{0,1}T^*M\right)^q$ is the bundle of (p,q)-covectors.

If $\alpha \in C^\infty(M, \Lambda^{0,q})$, then

$$d\alpha = \partial\alpha + \overline{\partial}\alpha,$$

where $\partial\alpha$ is a $(1,q)$-form and $\overline{\partial}\alpha$ is a $(0,q+1)$-form. Furthermore, the integrability of $T_{1,0}$ implies that $\overline{\partial}^2 = 0$. We thus get a chain complex,

$$\overline{\partial} : C^\infty(M, \Lambda^{0,\bullet}T^*M) \longrightarrow C^\infty(M, \Lambda^{0,\bullet+1}T^*M).$$

This complex is called the *Dolbeault complex* of M and its cohomology groups are denoted $H^{0,q}(M)$, $q = 0, \ldots, n$.

Let W be a holomorphic vector bundle over M. This means that M can be covered by open subsets U_α over which there are local trivializations $\tau_\alpha : W_{|U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^r$ in such way that the transition maps are holomorphic maps. Then, for $q = 0, \ldots, n$, there exists a unique operator

$$\overline{\partial}_W : C^\infty(M, (\Lambda^{0,q}T^*M) \otimes W) \longrightarrow C^\infty(M, (\Lambda^{0,q+1}T^*M) \otimes W)$$

such that, for all local holomorphic frames $e_1, \ldots, e_r$ of W and sections $\omega = \sum \omega_j \otimes e_j$ of $(\Lambda^{0,q}T^*M) \otimes W$,

$$\overline{\partial}_W\left(\sum \omega_j \otimes e_j\right) = \sum_j (\overline{\partial}\omega_j) \otimes e_j.$$

Since $(\overline{\partial}_W)^2 = 0$ this gives rise to a chain complex, called the *Dolbeault complex* of M *with coefficients in* W. Its cohomology groups are denoted $H^{0,q}(M, W)$, $q = 0, \ldots, n$.

Let us also endow W with a Hermitian metric $\langle ., . \rangle_W$. Together with the Hermitian metric on $\Lambda^{0,*}T^*M$ this defines a Hermitian metric on $(\Lambda^{0,*}T^*M) \otimes W$ and an inner inner products on $C^\infty(M, (\Lambda^{0,*}T^*M) \otimes W)$, using which we define the formal adjoint $\overline{\partial}_W^* : C^\infty(M, (\Lambda^{0,*}T^*M) \otimes W) \to C^\infty(M, (\Lambda^{0,*}T^*M) \otimes W)$. The *Dolbeault Laplacian* then is

$$\square_W := (\overline{\partial}_W + \overline{\partial}_W^*)^2 = \overline{\partial}_W^* \overline{\partial}_W + \overline{\partial}_W \overline{\partial}_W^*.$$

Notice that $\square_W$ maps $C^\infty(M, (\Lambda^{0,q}T^*M) \otimes W)$ to itself. We then denote by $\square_W^{0,q}$ its restriction to $(\Lambda^{0,q}T^*M) \otimes W$.

In addition, by looking at its expression in local coordinates, it is not difficult to check that $\square_W$ has same principal symbol as a Laplacian and hence is elliptic. As by assumption M is compact, it follows that $\ker \square_W$ is a finite dimensional subspace of $C^\infty(M, (\Lambda^{0,*}T^*M) \otimes W)$. We then define the *holomorphic Euler-characteristic* of M with coefficients in W to be

$$\chi(M, W) := \sum_{q=0}^{n} (-1)^q \dim H^{0,q}(M, W).$$

This is a holomorphic invariant of M and W.

PROPOSITION 8.11.15 (Hodge, see [**BGV**]). *For* $q = 0, \ldots, n$,

$$H^{0,q}(M, W) \simeq \ker \square_W^{0,q}.$$

Since the operators $\overline{\partial}_W + \overline{\partial}_W^*$ and $\square_W$ have same kernel, it follows from Proposition 8.11.15 that $H^{0,q}(M, W)$ is isomorphic to the kernel of the Dolbeault operator $\overline{\partial}_W + \overline{\partial}_W^*$ on $(0, q)$-forms. Thus,

$$\chi(M, W) = \operatorname{ind} \left(\overline{\partial}_W + \overline{\partial}_W^* \right), \tag{8.9}$$

where the index of $\overline{\partial}_W + \overline{\partial}_W^*$ is defined to be

$$\dim \ker \left(\overline{\partial}_W + \overline{\partial}_W^* \right)_{|(\Lambda^{0,\mathrm{ev}}T^*M)\otimes W} - \dim \ker \left(\overline{\partial}_W + \overline{\partial}_W^* \right)_{|(\Lambda^{0,\mathrm{odd}}T^*M)\otimes W}.$$

Let $\nabla : C^\infty(M, W) \to C^\infty(M, T_{\mathbb{C}}^*M \otimes W)$ be a connection on W. Using the splitting $T_{\mathbb{C}}^*M = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M$ we can split ∇ as

$$\nabla = \nabla^{1,0} + \nabla^{0,1},$$

where $\nabla^{1,0}$ (resp., $\nabla^{0,1}$) maps to sections of $(\Lambda^{1,0}T^*M) \otimes W$ (resp., sections of $(\Lambda^{0,1}T^*M) \otimes W$). We then say that ∇ is a *holomorphic connection* when $\nabla^{0,1} = \overline{\partial}_W$. If ∇ is a holomorphic connection, then its curvature F^W is actually an $(1,1)$-form with values in $\operatorname{End} W$.

PROPOSITION 8.11.16. *There is a unique holomorphic connection* ∇^W *on* W *that preserves its Hermitian metric, i.e.,*

$$\langle \nabla_X^W w_1, w_2 \rangle_W + \langle w_1, \nabla_X^W w_2 \rangle = X(\langle w_1, w_2 \rangle_W) \quad \forall w_j \in C^\infty(M, W) \quad \forall X \in C^\infty(M, TM).$$

The connection ∇^W in Proposition 8.11.16 is called the *Chern connection* of W.

Let Ω be the imaginary part of the Hermitian metric h of TM. This is a 2-form on M since, for all $X, Y \in C^\infty(M, TM)$,

$$\Omega(Y, X) = \Im h(Y, X) = \Im \overline{h(X, Y)} = -\Im h(X, Y) = -\Omega(X, Y).$$

We then say that M is a *Kähler manifold* when Ω is a closed form, i.e., $d\Omega = 0$.

Recall that by assumption the Riemannian metric of M is the real part of the Hermitian metric h.

Proposition 8.11.17 (See [**BGV**]). *The manifold M is Kähler if and only if the Levi-Civita connection ∇^{TM} on TM is a holomorphic connection, i.e., it agrees with the Chern connection of TM.*

From now on, we assume that M is a compact Kähler manifold. Then ∇^{TM} preserves the complex structure, and hence it can be lifted to a Hermitian connection on $\Lambda^{0,q}T^*M$ for each $q = 0, 1, \ldots, n$.

In addition, in the very same way as we constructed the spinor representation, we can endow the bundle $\Lambda^{0,*}T^*M$ with a Clifford-module bundle structure as follows.

The $\mathbb{Z}_2$-grading of $\Lambda^{0,*}T^*M$ is simply given by the splitting,

$$\Lambda^{0,*}T^*M = \Lambda^{0,\mathrm{ev}}T^*M \oplus \Lambda^{0,\mathrm{odd}}T^*M.$$

The action of $\mathrm{Cl}_{\mathbb{C}}(M)$ is such that, above all $x \in M$,

$$c(\xi).\omega = \sqrt{2}\left(\varepsilon(\xi^{1,0}) - \imath(\xi^{1,0})\right)\omega \qquad \forall \xi \in T^*_{\mathbb{C},x}M \quad \forall \omega \in \Lambda^{0,*}T^*M,$$

where $\varepsilon(\xi^{1,0})$ is the exterior by the component $\xi^{0,1}$ of ξ in $\Lambda^{0,1}T^*_xM$ and $\imath(\xi^{1,0})$ is the interior product by its component $\xi^{1,0}$ in $\Lambda^{1,0}T^*_xM$.

This allows us to realize the spinor bundle of M as the bundle $\Lambda^{0,*}T^*M$ equipped with the above Clifford-module structure. We then endow the bundle $(\Lambda^{0,*}T^*M) \otimes W$ with its twisted Clifford-module bundle structure. Namely, its Z_2-grading is given by

$$(\Lambda^{0,*}T^*M) \otimes W = \Big((\Lambda^{0,\mathrm{ev}}T^*M) \otimes W\Big) \oplus \Big((\Lambda^{0,*}T^*M) \otimes W\Big),$$

and the action of $\mathrm{Cl}_{\mathbb{C}}(M)$ is such that, for all $x \in M$,

$$a.(\omega \otimes w) = (a.\omega) \otimes w \qquad \forall a \in \mathrm{Cl}_x(M) \quad \forall (\omega, w) \in \Lambda^{0,*}T^*_xM \times W_x.$$

In addition, we equipped $(\Lambda^{0,*}T^*M) \otimes W$ with the twisted connection,

$$\nabla^{(\Lambda^{0,*}T^*M)\otimes W} := \nabla^{\Lambda^{0,*}T^*M} \otimes 1 + 1 \otimes \nabla^W,$$

where ∇^W is the Chern connection of W.

Lemma 8.11.18 (See [**BGV**]). *If M is Kähler, then $\nabla^{(\Lambda^{0,*}T^*M)\otimes W}$ is a Clifford connection on the Clifford-module bundle $(\Lambda^{0,*}T^*M) \otimes W$ and its associated Dirac operator is equal to $\sqrt{2}(\overline{\partial}_W + \overline{\partial}_W^*)$.*

In the sequel, we denote by $R^{T^{1,0}M}$ the curvature of the Levi-Civita connection of $T^{1,0}M$.

Lemma 8.11.19 (See [**BGV**]).

(1) *The twisted curvature of $\nabla^{(\Lambda^{0,*}T^*M)\otimes W}$ is given by*

$$F^{(\Lambda^{0,*}T^*M)\otimes W/\$} = \frac{1}{2}\,\mathrm{Tr}\left[R^{T^{1,0}M}\right] + 1 \otimes F^W. \tag{8.10}$$

(2) *We have*

$$\hat{A}(R^M) = \det\left[\frac{R^{T^{1,0}M}}{e^{\frac{1}{2}R^{T^{1,0}M}} - e^{-\frac{1}{2}R^{T^{1,0}M}}}\right]. \tag{8.11}$$

It follows from (8.10) that the relative Chern form of $F^{(\Lambda^{0,*}T^*M)\otimes W/\not{S}}$ equals

$$\exp\left(-\frac{1}{2}\operatorname{Tr}\left[R^{T^{1,0}M}\right]\right)\wedge \operatorname{Ch}(F^W) = \det\left[e^{-\frac{1}{2}R^{T^{1,0}M}}\right]\wedge \operatorname{Ch}(F^W).$$

Combining this with (8.11) shows that $\hat{A}(R^M)\wedge \operatorname{Ch}(F^{(\Lambda^{0,*}T^*M)\otimes W/\not{S}})$ is equal to

$$\det\left[\frac{R^{T^{1,0}M}}{e^{\frac{1}{2}R^{T^{1,0}M}}-e^{-\frac{1}{2}R^{T^{1,0}M}}}\right]\wedge\det\left[e^{-\frac{1}{2}R^{T^{1,0}M}}\right]\wedge\operatorname{Ch}(F^W) = \operatorname{Td}(R^{T^{1,0}M})\wedge\operatorname{Ch}(F^W),$$

where $\operatorname{Td}(R^{T^{1,0}M})$ is the Todd form of $R^{T^{1,0}M}$, i.e.,

$$\operatorname{Td}(R^{T^{1,0}M}) = \det\left[\frac{R^{T^{1,0}M}}{e^{R^{T^{1,0}M}}-1}\right].$$

Therefore, by applying the local index formula for Atiyah-Singer and using (8.9) we obtain

Theorem 8.11.20 (Hirzebruch-Riemann-Roch). *If M is Kähler, then*

$$\chi(M,W) = \operatorname{ind}(\overline{\partial}_W + \overline{\partial}_W^*) = (2i\pi)^{-\frac{n}{2}}\int_M \left[\operatorname{Td}(R^{T^{1,0}M})\wedge\operatorname{Ch}(F^W)\right]^{(n)}.$$

8.12. Proof of the Local Index Theorem

In this section we reproduce the proof of the local index theorem (i.e., Theorem 8.10.5) given in [**Po**]. As mentioned in Section 8.10, this theorem yields the local index formula of Atiyah-Singer. The proof here is given for twisted Dirac operators on spin manifolds, but the argument can be extended to more general Dirac operators on Clifford-module bundle coming from a Clifford connection.

The argument is based on combining the rescaling of Getzler [**Ge2**] with the approach to the heat kernel asymptotics of Greiner [**Gr**].

8.12.1. Greiner's approach of the heat kernel asymptotics. In this section we recall Greiner's approach of the heat kernel asymptotics as in [**Gr**] and [**BGS**].

Let E be a Hermitian vector bundle over M and let $\Delta : C^\infty(M,E)\rightarrow C^\infty(M,E)$ be a selfadjoint elliptic second order differential operator with positive principal symbol.

If we regard Δ as an unbounded operator on $L^2(M,E)$ with domain the Sobolev space $L^2_2(M,E)$, then Δ is selfadjoint and bounded from below. Therefore, by standard Borel functional calculus we can define the heat semi-group $e^{-t\Delta}$, $t\geq 0$, as a family of selfadjoint bounded operators on $L^2(M,E)$.

Furthermore, the ellipticity of Δ implies that, for all $t>0$, the operator $e^{-t\Delta}$ is smoothing. That is, its Schwartz kernel $k_t(x,y)$ is contained in $C^\infty(M,E)\hat{\otimes}C^\infty(M,E^*\otimes|\Lambda|(M))$, where $|\Lambda|(M)$ is the density bundle of M. The kernel $k_t(x,y)$ is called the *heat kernel* of Δ.

Recall that the heat semigroup allows us to invert the heat equation. Namely, consider the operator $Q_0 : C_c^\infty(M\times\mathbb{R},E)\rightarrow \mathcal{D}'(M\times\mathbb{R},E)$ defined by

$$Q_0u(x,s) := \int_0^\infty e^{-s\Delta}u(x,t-s)dt \qquad \forall u\in C_c^\infty(M\times\mathbb{R},E). \tag{8.12}$$

Then Q_0 maps continuously $C_c^\infty(M \times \mathbb{R}, E)$ to $C^0(\mathbb{R}, L^2(M, E)) \subset \mathcal{D}'(M \times \mathbb{R}, E)$ and satisfies

$$(\Delta + \partial_t)Q_0 u = Q_0(\Delta + \partial_t)u = u \qquad \forall u \in C_c^\infty(M \times \mathbb{R}, E).$$

Notice that the operator Q_0 has the *Volterra property* in the sense of [**Pi**], i.e., it has a Schwartz kernel of the form $K_{Q_0}(x, y, t - s)$, where $K_{Q_0}(x, y, t)$ vanishes on the region $t < 0$. In fact,

$$K_{Q_0}(x, y, t) = \begin{cases} k_t(x, y) & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

These equalities are the main motivation for using pseudodifferential techniques to study the heat kernel $k_t(x, y)$. The idea is to consider a class of ΨDOs, the Volterra ΨDOs([**Gr**], [**Pi**], [**BGS**]), taking into account

(i) The aforementioned Volterra property.
(ii) The parabolic homogeneity of the heat operator $\Delta + \partial_t$, i.e., the homogeneity with respect to the dilations $\lambda.(\xi, \tau) = (\lambda\xi, \lambda^2\tau)$, $(\xi, \tau) \in \mathbb{R}^{n+1}$, $\lambda \neq 0$.

In the sequel, for $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we denote by g_λ the element of $\mathcal{S}'(\mathbb{R}^{n+1})$ defined by

$$\langle g_\lambda(\xi, \tau), u(\xi, \tau)\rangle := |\lambda|^{-(n+2)}\langle g(\xi, \tau), u(\lambda^{-1}\xi, \lambda^{-2}\tau)\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^{n+1}). \tag{8.13}$$

Definition 8.12.1. *A distribution $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ is (parabolic) homogeneous of degree m, $m \in \mathbb{Z}$, when*

$$g_\lambda = \lambda^m g \qquad \forall \lambda \in \mathbb{R} \setminus 0.$$

In the sequel, we denote by $\mathbb{C}_-$ the complex halfplane $\{\Im\tau > 0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

Lemma 8.12.2 ([**BGS**, Prop. 1.9]). *Let $q(\xi, \tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R}) \setminus 0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) *$q(x, \xi, \tau)$ extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}_-}) \setminus 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.*

Then there is a unique $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1} \setminus 0$ so that:

(ii) *g is homogeneous of degree m.*
(iii) *The inverse Fourier transform $\check{g}(x, t)$ vanishes for $t < 0$.*

Remark 8.12.3. If $m \leq -(n + 2)$, then (ii) need not hold for symbols that do not satisfy (i).

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOson $U \times \mathbb{R}^{n+1} \setminus 0$ as follows.

Definition 8.12.4. *$S_{\mathrm{v}}^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q(x, \xi, \tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q(x, \xi, \tau) \sim \sum_{j \geq 0} q_{m-j}(x, \xi, \tau)$, where*

- *$q_l(x, \xi, \tau) \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R}) \setminus 0])$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 8.12.2 with respect to the last $n + 1$ variables.*

- *The sign $\sim$ means that, for all compacts $K \subset U$, integers N and k and multi-orders α and β, there is a constant $C_{NK\alpha\beta k} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{1/2})^{m-N-|\beta|-2k}, \tag{8.14}$$

for all $(x,\xi,\tau) \in K \times \mathbb{R}^n \times \mathbb{R}$ with $|\xi| + |\tau|^{\frac{1}{2}} \geq 1$.

In the sequel, for any symbol $q(x,\xi,\tau) \in S_{\mathrm{v}}^m(U \times \mathbb{R})$ we denote by $q(x, D_x, D_t)$ the operator from $C_c^\infty(U \times \mathbb{R})$ to $C^\infty(U \times \mathbb{R})$ defined by

$$q(x, D_x, D_t)u(x,t) := (2\pi)^{-(n+1)} \iint e^{i(x\cdot\xi + t\tau)} q(x,\xi,\tau)\hat{u}(\xi,\tau) d\xi d\tau \quad \forall u \in C_c^\infty(U \times \mathbb{R}).$$

Definition 8.12.5. *$\Psi_{\mathrm{v}}^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists of continuous linear operators $Q : C_c^\infty(U_x \times \mathbb{R}_t) \to C^\infty(U_x \times \mathbb{R}_t)$ such that*

(i) *Q has the Volterra property.*
(ii) *Q can be put in the form,*

$$Q = q(x, D_x, D_t) + R,$$

for some symbol $q(x,\xi,\tau) \in S_{\mathrm{v}}^m(U \times \mathbb{R})$ and some smoothing operator R.

If Q is a Volterra ΨDO we shall denote by $K_Q(x,y,t-s)$ its distribution kernel, so that the distribution $K_Q(x,y,t)$ vanishes for $t < 0$.

Example 8.12.6. Let P be a differential operator of order 2 on U and let $p_2(x,\xi)$ denote the principal symbol of P. Then the heat operator $P + \partial_t$ is a Volterra ΨDO of order 2 with principal symbol $p_2(x,\xi) + i\tau$.

Other examples of Volterra ΨDOsare given by the homogeneous operators defined below.

Definition 8.12.7. *Let $q_m(x,\xi,\tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order m and let $g_m \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 8.12.2. Then:*

- *$\check{q}_m(x,y,t)$ is the inverse Fourier transform of $g_m(x,\xi,\tau)$ in the last $n+1$ variables.*
- *$q_m(x, D_x, D_t)$ is the operator with kernel $\check{q}_m(x, y-x, t)$.*

Proposition 8.12.8 ([**Gr**], [**Pi**], [**BGS**]). *The following hold.*

(1) Composition. *Let $Q_j \in \Psi_{\mathrm{v}}^{m_j}(U \times \mathbb{R})$, $j = 1, 2$, have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then Q_1Q_2 lies in $\Psi_{\mathrm{v}}^{m_1+m_2}(U \times \mathbb{R})$ and has symbol $q_1 \# q_2 \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_\xi^\alpha q_2$.*
(2) Parametrices. *An operator $Q \in \Psi_{\mathrm{v}}^m(U \times \mathbb{R})$ admits a parametrix in $\Psi_{\mathrm{v}}^{-m}(U \times \mathbb{R})$ if, and only if, its principal symbol is nowhere vanishing on $U \times [(\mathbb{R}^n \times \overline{\mathbb{C}_-} \setminus 0)]$.*
(3) Invariance. *Let $\phi : U \to V$ be a diffeomorphism onto another open subset V of $\mathbb{R}^n$ and let Q be a Volterra Ψ on $U \times \mathbb{R}$ of order m. Then $Q = (\phi \oplus \mathrm{id}_{\mathbb{R}})_* Q$ is a Volterra Ψ on $V \times \mathbb{R}$ of order m.*

In addition, the following property shows the relevance of Volterra ΨDOs for deriving small times asymptotics.

LEMMA 8.12.9 ([**Gr**, Chap. I], [**BGS**, Thm. 4.5]). *Let $Q \in \Psi_{\mathrm{v}}^m(U \times \mathbb{R})$ have symbol $q \sim \sum q_{m-j}$. Then ,in $C^\infty(U)$,*

$$K_Q(x,y,t) \sim t^{-(\frac{n}{2}+[\frac{m}{2}]+1)} \sum_{l \geq 0} t^l \check{q}_{2[\frac{m}{2}]-2l}(x,0,1) \quad \textit{as } t \to 0^+, \tag{8.15}$$

where the notation $\check{q}_k$ has the same meaning as in Definition 8.12.7.

PROOF. As the Fourier transform relates the decay at infinity to the behavior at the origin of the Fourier transform the distribution $\check{q}-\sum_{j\leq J} \check{q}_{m-j}$ lies in $C^N(U_x \times \mathbb{R}^n_y \times \mathbb{R}_t)$ as soon as J is large enough. Since $Q-q(x,D_x,D_t)$ is smoothing it follows that $R_J(x,t) := K_Q(x,x,t) - \sum_{j\leq J} \check{q}_{m-j}(x,0,t)$ is of class C^N. As $R_J(x,y,t)=0$ for $t<0$ we see that

$$\partial_t^l R_J(x,0) = 0 \qquad \text{for } l = 0,1,\ldots,N.$$

Thus $R_J(.,t) = \mathrm{O}(t^N)$ in $C^N(U)$ as $t \to 0^+$. This proves that, in $C^\infty(U)$,

$$K_Q(x,x,t) \sim \sum \check{q}_{m-j}(x,0,t) \quad \text{as } t \to 0^+. \tag{8.16}$$

Let $j \in \mathbb{N}_0$. Observe that, for all $\lambda \neq 0$,

$$(\check{q}_{m-j})_\lambda = |\lambda|^{-(n+2)}(q_{m-j,\lambda^{-1}})^\vee = |\lambda|^{-(n+2)}\lambda^{j-m}\check{q}_{m-j}.$$

Thus, setting $\lambda = \sqrt{t}$ with $t>0$, we get

$$\check{q}_{m-j}(x,0,t) = t^{\frac{j-n-m}{2}-1}\check{q}_{m-j}(x,0,1).$$

Furthermore, if we take $\lambda=-1$ and $m-j$ is odd, then

$$\check{q}_{m-j}(x,0,1) = (-1)^{m-j}q_{m-j}(x,0,1) = -q_{m-j}(x,0,1) = 0.$$

Combining all this with (8.16) shows that, in $C^\infty(U)$,

$$K_Q(x,x,t) \sim_{t\to 0^+} \sum_{m-j \text{ even}} t^{\frac{j-n-m}{2}-1}\check{q}_{m-j}(x,0,1), \tag{8.17}$$

that is, (8.15) holds. The lemma is thus proved. □

The invariance property in Proposition 8.12.8 allows us to define Volterra ΨDOs on $M \times \mathbb{R}$ acting on the sections of the vector bundle E. Then all the preceding properties hold *verbatim* in this context. In particular, the heat operator $\Delta + \partial_t$ has a parametrix Q in $\Psi_{\mathrm{v}}^{-2}(M, \times\mathbb{R}, E)$.

In fact, comparing the operator (8.12) with any Volterra parametrix for $\Delta+\partial_t$ allows us to prove

THEOREM 8.12.10 ([**Gr**], [**Pi**], [**BGS**, pp. 363-362]). *The differential operator $\Delta + \partial_t$ is invertible and its inverse $(\Delta+\partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

Combining this with Lemma 8.12.9 gives the heat kernel asymptotics below.

THEOREM 8.12.11 ([**Gr**, Thm. 1.6.1]). *In $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$ we have*

$$k_t(x,x) \sim_{t\to 0^+} t^{\frac{-n}{2}} \sum_{l\geq 0} t^l a_l(\Delta)(x), \qquad a_l(\Delta)(x) = \check{q}_{-2-2l}(x,0,1), \tag{8.18}$$

where the equality on the right-hand side shows how to compute the densities $a_l(\Delta)(x)$ in local trivializing coordinates by means of the symbol $q(x,\xi,\tau) \sim \sum q_{-2-j}(x,\xi,\tau)$ of any Volterra parametrix for $\Delta + \partial_t$.

This approach to the heat kernel asymptotics present several advantages. First, as Theorem 8.12.11 is a purely local statement we can easily localize the heat kernel asymptotics. In fact, given a Volterra parametrix Q for $\Delta + \partial_t$ in some local trivializing coordinates around $x_0 \in M$, comparing the asymptotics (8.15) and (8.18) we get

$$k_t(x_0, x_0) = K_Q(x_0, x_0, t) + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{8.19}$$

Therefore in order to determine the heat kernel asymptotics (8.18) at x_0 we only need a Volterra parametrix for $\Delta + \partial_t$ near x_0.

Second, we have a genuine asymptotics with respect to the C^∞-topology and it can be differentiated as follows.

Proposition 8.12.12. *Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a differential operator of order m and let us denote by $h_t(x, y)$ the Schwartz kernel of $Pe^{-t\Delta}$. Then, in $C^\infty(M, |\Lambda| \otimes \operatorname{End} E)$,*

$$h_t(x, x) \sim_{t \to 0^+} t^{[\frac{m}{2}] - \frac{n}{2}} \sum_{l \geq 0} t^l b_l(x), \qquad b_l(x) = \check{r}_{2[\frac{m}{2}]-2-2l}(x, 0, 1), \tag{8.20}$$

where the equality on the right-hand side gives a formula for computing the densities $b_l(x)$ in local trivializing coordinates by means of the symbol $r \sim \sum r_{m-2-j}$ of $R = P(\Delta + \partial_t)^{-1}$ (or of $R = PQ$, where Q is any Volterra parametrix for $\Delta + \partial_t$).

Proof. Observe that

$$h_t(x, y) = P_x k_t(x, y) = P_x K_{(\Delta+\partial_t)^{-1}}(x, y, t) = K_{P(\Delta+\partial_t)^{-1}}(x, y, t).$$

Therefore, the result follows by applying Lemma 8.12.9 to $P(\Delta + \partial_t)^{-1}$ (or to PQ, where Q is any Volterra parametrix for $\Delta + \partial_t$). $\square$

Finally, in local trivializing coordinates the densities $a_j(\Delta)(x)$'s can be explicitly computed in terms of the symbol $p = p_2 + p_1 + p_0$ of Δ. To see this let $q \sim \sum q_{-2-j}$ be the symbol of a Volterra parametrix Q for $\Delta + \partial_t$. As $q \# p \sim q(p + i\tau) + \sum \frac{1}{\alpha!} \partial_\xi^\alpha q D_x^\alpha p \sim 1$ we get $q_{-2} = (p_2 + i\tau)^{-1}$ and

$$q_{-2-j} = -\Big(\sum_{k+l+|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha q_{-2-k} D_x^\alpha p_{2-l}\Big)(p_2 + i\tau)^{-1}, \quad j \geq 1. \tag{8.21}$$

Therefore, combining with (8.18) we deduce that, as in [**Gi**], the densities $a_j(\Delta)(x)$'s are universal polynomials in the the jets at x_0 of the symbol of Δ with coefficients depending smoothly on its principal symbol.

Similarly, in local trivializing coordinates the densities $b_l(x)$'s in (8.20) can be expressed universal polynomials in the the jets at x_0 of the symbols of Δ and P with coefficients depending smoothly on the principal symbol of Δ.

8.12.2. Proof of the Local Index Theorem. In this subsection, we shall prove the local index theorem. We shall use the same notation as in Section 8.10. In particular, E is a Clifford-module bundle with Clifford connection ∇^E and associated Dirac operator D_E. In addition, we denote by $k_t(x, y)$ the heat kernel of D_E^2.

It follows from Theorem 8.12.11 that, in $C^\infty(M, |\Lambda|(M))$,

$$\operatorname{Str}_E[k_t(x, x)] \sim t^{-\frac{n}{2}} \sum_{j \geq 0} t^j \operatorname{Str}_E[a_j(D_E^2)(x)] \qquad \text{as } t \to 0^+.$$

We thus have an asymptotics in $C^\infty(M, |\Lambda|(M))$. Therefore, in order to prove the local index theorem it is enough to show that, near any given point x_0, in some local coordinates centered at x_0 and local trivialization of E near $x_0 E$,

$$\operatorname{Str}_E[k_t(0,0)] = (2i\pi)^{-\frac{n}{2}} \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^{E/\$})\right]^{(n)}(0) + \mathrm{O}(t). \tag{8.22}$$

Furthermore, if follows from (8.19) that in order to so we may replace the heat kernel $k_t(x,x)$ by the kernel $K_Q(x,y,t)$ of any Volterra-ΨDO parametrix for $D_E^2 + \partial_t$ defined near $x = 0$. Indeed, we then have

$$k_t(0,0) = K_Q(0,0,t) + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{8.23}$$

This enables use to reduce to the case where M is a neighborhood of the origin in $\mathbb{R}^n$ and E is a trivial vector bundle.

As we have total freedom on the choice of the local coordinates and the trivializations of E, we may choose to use local Riemannian coordinates and introduce a synchronous orthonormal tangent frame $\{e_1, \cdots, e_n\}$ such that $e_j = \partial_j$. Using this frame to construct the spinor representation, near $x = 0$ we may realize E as a trivial twisted Clifford bundle with fiber $\S_n \otimes W$, where $\$_n$ is the spinor space of $\mathbb{R}^n$ and W is a $\mathbb{Z}_2$-graded vector space. Notice that the Clifford connection ∇^E then is a twisted Clifford connection $\nabla^{\$_n} \otimes 1 + 1 \otimes \nabla^W$, where $\nabla^{\$_n}$ is the spin connection on the trivial vector bundle $M \times \S_n$ and ∇^W is a Hermitian connection on $M \times W$. Incidentally, if we denote by F^W the curvature of ∇^W, then the twisted curvature $F^{E/\$}$ and its relative Chern form are given by

$$F^{E/\$} = 1 \otimes F^W \qquad \text{and} \qquad \operatorname{Ch}(F^{E/\$}) = \operatorname{Ch}(F^W).$$

In addition, as we are using normal coordinates, near $x = 0$ the coefficients $g_{ij} = g(\partial_i, \partial_j)$ of the metric g are such that

$$g_{ij}(x) = \delta_{ij} + \mathrm{O}(|x|^2), \qquad \omega_{ikl}(x) = -\frac{1}{2} R^M_{ijkl}(0) x^j + \mathrm{O}(|x|^2), \tag{8.24}$$

A proof of this asymptotics can be found for instance in [**BGV**].

Let us also introduce the coefficients of the Levi-Civita connection ∇^{TM} and the curvature tensor R^M defined in terms of the synchronous orthonormal tangent frame $\{e_i\}$ by

$$\omega_{ikl} = \langle \nabla_i^{TM} e_k, e_l \rangle \quad \text{and} \quad R^M_{ijkl} = \langle R^M(\partial_i, \partial_j) e_k, e_l \rangle.$$

Then, near $x = 0$,

$$\omega_{ikl}(x) = -\frac{1}{2} R^M_{ijkl}(0) x^j + \mathrm{O}(|x|^2). \tag{8.25}$$

See, e.g., [**BGV**] for a proof of this asymptotics.

Recall that the quantification and symbol maps are linear isomorphisms,

$$\Lambda^*_{\mathbb{C}} \mathbb{R}^n \xrightarrow{c} \operatorname{Cl}_{\mathbb{C}}(\mathbb{R}^n) \quad \text{and} \quad \operatorname{Cl}_{\mathbb{C}}(\mathbb{R}^n) \xrightarrow{\sigma} \Lambda^*_{\mathbb{C}} \mathbb{R}^n. \tag{8.26}$$

As n is even the spinor representation $\rho : \operatorname{Cl}_{\mathbb{C}}(\mathbb{R}^n) \to \operatorname{End} \$_n$ is an algebra isomorphism which allows us to identify $\operatorname{Cl}_{\mathbb{C}}(\mathbb{R}^n)$ with $\operatorname{End} \$_n$. Bearing in mind this identification, we shall also denote by c and σ the linear isomorphisms,

$$\Lambda^*_{\mathbb{C}} \mathbb{R}^n \xrightarrow{c} \operatorname{End} \$_n \quad \text{and} \quad \operatorname{End} \$_n \xrightarrow{\sigma} \Lambda^*_{\mathbb{C}} \mathbb{R}^n, \tag{8.27}$$

which are obtained by composing the linear isomorphisms (8.26) with ρ or its inverse.

As E is a trivial bundle with fiber $\$_n \otimes W$, we can regard the Volterra ΨDOs on $M \times \mathbb{R}$ acting on the sections of E as elements of $\Psi_{\mathrm{v}}^*(M \times \mathbb{R}) \otimes (\operatorname{End} \$_n) \otimes (\operatorname{End} W)$. Using the linear maps (8.27), we then get linear maps,

$$\Psi_{\mathrm{v}}^*(M \times \mathbb{R}) \otimes (\Lambda_{\mathbb{C}}^* \mathbb{R}^n) \otimes (\operatorname{End} W) \xrightarrow{c} \Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E),$$

$$\Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E) \xrightarrow{\sigma} \Psi_{\mathrm{v}}^*(M \times \mathbb{R}) \otimes (\Lambda_{\mathbb{C}}^* \mathbb{R}^n) \otimes (\operatorname{End} W).$$

We get similar linear maps at the level of symbols and Schwartz kernels.

As E is the trivial $\mathbb{Z}_2$-graded bundle with fiber $\$_n \otimes W$, its supertrace is just $\operatorname{Str}_{\$_n} \otimes \operatorname{Str}_W$, where $\operatorname{Str}_{\$_n}$ is the supertrace on the trivial bundle $M \times \$_n$ and Str_W is the supertrace on W. Notice that, as the metric on M varies on the fibers of TM, so does the supertrace on the fibers of $M \times \operatorname{End}(\$_n)$. However, thanks (8.24) at $x = 0$ the metric $g(x)$ agrees with the standard Euclidean metric, and hence we may use Proposition 4.8 to get

$$\operatorname{Str}_{\$_n}[T(0)] = (-2i)^{\frac{n}{2}} \sigma[T(0)]^{(n)} \qquad \forall T \in C^\infty(M, \operatorname{End} \$_n),$$

where $\sigma[T(0)]^{(n)}$ is the n-the degree component of $\sigma[T(0)] \in \Lambda_{\mathbb{C}}^* \mathbb{R}^n$.

It follows from all this that, for all $P \in \Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E)$,

$$\operatorname{Str}_E [K_P(0,0,t)] = (-2i)^{\frac{n}{2}} (\sigma \otimes \operatorname{Str}_W)[K_P(0,0,t)]^{(n)} \qquad \forall t > 0.$$

Combining this with (8.23) then shows that

$$k_t(0,0) = (-2i)^{\frac{n}{2}} (\sigma \otimes \operatorname{Str}_W)[K_P(0,0,t)]^{(n)} + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{8.28}$$

Therefore, the proof of (8.22) boils down to showing the existence of a small-time limit of $(\sigma \otimes \operatorname{Str}_W)[K_P(0,0,t)]^{(n)}$ and identifying it.

In order to study the small-time behavior of $(\sigma \otimes \operatorname{Str}_W)[K_P(0,0,t)]^{(n)}$ we shall implement the rescaling of Getzler [**Ge2**] as a filtration on $\Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E)$. Roughly speaking this rescaling aims at assigning the following degrees:

$$\deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx^j) = -\deg x^j = 1, \tag{8.29}$$

while $\deg B = 0$ for any $B \in \operatorname{End} W$. We then obtain a filtration on $\Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E)$ as follows.

Let $Q \in \Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E)$ have symbol $q(x, \xi, \tau) \sim \sum_{k \leq m'} q_k(x, \xi, \tau)$. Then taking components in each subspace $\Lambda^j T_{\mathbb{C}}^* \mathbb{R}^n(n)$ and then using Taylor expansions at $x = 0$ gives formal expansions

$$\sigma[q(x, \xi, \tau)] \sim \sum_{j,k} \sigma[q_k(x, \xi, \tau)]^{(j)} \sim \sum_{j,k,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_k(0, \xi, \tau)]^{(j)}. \tag{8.30}$$

According to (8.29) the symbol $\frac{x^\alpha}{\alpha!} \partial_x^\alpha \sigma[q_k(0, \xi, \tau)]^{(j)}$ is Getzler homogeneous of degree $k + j - |\alpha|$. Therefore, we can expand $\sigma[q(x, \xi, \tau)]$ as

$$\sigma[q(x, \xi, \tau)] \sim \sum_{j \geq 0} q_{(m-j)}(x, \xi, \tau), \qquad q_{(m)} \neq 0, \tag{8.31}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m - j$.

Definition 8.12.13. *Using (8.31) we make the following definitions:*

- *The integer m is the Getzler order of Q.*
- *The symbol $q_{(m)}$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is the principal Getzler homogeneous symbol of Q.*

- *The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ on $\mathbb{R}^n \times \mathbb{R}$ is the model operator of Q.*

REMARK 8.12.14. The model operator $Q_{(m)}$ is well defined according to definition 8.12.7.

REMARK 8.12.15. By construction we always have Getzler order $\leq$ order $+ n$, but this is not an equality in general.

EXAMPLE 8.12.16. Let $A = A_i dx^i$ be the connection one-form on W. Then by (8.25) the covariant derivative $\nabla_i^E = \partial_i + \frac{1}{4}\omega_{ikl}(x)c(e^k)c(e^l) + A_i$ on $\$_n \otimes W$ has Getzler order 1 and model operator

$$\nabla^E_{i(1)} = \partial_i - \frac{1}{4}R^M_{ij}(0)x^j, \qquad R^M_{ij}(0) := R^M_{ijkl}(0)dx^k \wedge dx^l. \tag{8.32}$$

The interest to introduce Getzler orders stems from the following.

LEMMA 8.12.17. *Let $Q \in \Psi_{\mathrm{v}}^*(M \times \mathbb{R}, M)$ have Getzler order m and model operator $Q_{(m)}$. Then, as $t \to 0^+$,*

$$\sigma[K_Q(0,0,t)]^{(j)} = \begin{cases} \mathrm{O}(t^{\frac{j-m-n-1}{2}}) & \text{if } m-j \text{ is odd,} \\ t^{\frac{j-m-n}{2}-1}K_{Q_{(m)}}(0,0,1)^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}}) & \text{if } m-j \text{ is even.} \end{cases}$$

In particular, if $m = -2$, then

$$\sigma[K_Q(0,0,t)]^{(n)} = K_{Q_{(-2)}}(0,0,1)^{(n)} + \mathrm{O}(t). \tag{8.33}$$

PROOF. Let $q(x,\xi,\tau) \sim \sum q_k(x,\xi,\tau)$ be the symbol of Q and let $q_{(m)}(x,\xi,\tau)$ be its principal Getzler-homogeneous symbol. By Lemma 8.12.9 we have

$$\sigma[K_Q(0,0,t)]^{(j)} \sim_{t\to 0^+} \sum t^{-\frac{n+2+m-j}{2}}\sigma[\check{q}_k(0,0,1)]^{(j)}, \tag{8.34}$$

and we know that $\check{q}_k(0,0,1) = 0$ if k is odd. Moreover, the symbol $\sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k+j$, and so it must be zero if $k + j > m$ since otherwise Q would not have Getzler order m. Therefore,

$$\sigma[K_Q(0,0,t)]^{(j)} = \begin{cases} \mathrm{O}(t^{\frac{j-m-n-1}{2}}) & \text{if } m-j \text{ is odd,} \\ t^{\frac{j-m-n}{2}-1}\sigma[\check{q}_{m-j}(0,0,1)]^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}}) & \text{if } m-j \text{ is even.} \end{cases} \tag{8.35}$$

In addition, the symbol $\sigma[q_{(m)}(0,\xi,\tau)]^{(j)}$ is equal to

$$\sum_{k+j-|\alpha|=m} \left(\frac{x^\alpha}{\alpha!}\partial_x^\alpha \sigma[q_k(0,\xi,\tau)]^{(j)}\right)_{x=0} = \sigma[q_{m-j}(0,\xi,\tau)]^{(j)}. \tag{8.36}$$

As $K_{Q_{(m)}}(x,y,t) = (q_{(m)})^\vee(x,y,t)$, we deduce that

$$\sigma[\check{q}_{m-j}(0,0,1)]^{(j)} = (q_{(m)})^\vee(0,0,1)^{(j)} = K_{Q_{(m)}}(0,0,1)^{(j)}.$$

Combining this with (8.35) proves the lemma. ☐

In the sequel, we say that a symbol or a ΨDO is $\mathrm{O}_{\mathrm{G}}(m)$ if it has Getzler order $\leq m$.

LEMMA 8.12.18. *For $j = 1, 2$ let $Q_j \in \Psi_{\mathrm{v}}^*(M \times \mathbb{R}, E)$ have Getzler order m_j and model operator $Q_{(m_j)}$. In addition, assume that Q_1 or Q_2 is properly supported. Then*

$$Q_1Q_2 = c\left[Q_{(m_1)}Q_{(m_2)}\right] + \mathrm{O}_G(m_1 + m_2 - 1). \tag{8.37}$$

PROOF. Let q_j be the symbol of Q_j and let $q_{(m_j)}$ be its principal Getzler homogeneous symbol. By Proposition 8.12.8 the operator Q_1Q_2 has symbol $q_1\#q_2$.

Moreover, for N large enough, the symbol $q_1\#q_2 - \sum_{|\alpha|\leq N}\frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_x^\alpha q_2$ has order $< m_1+m_2-n$, and hence has Getzler order $< m_1+m_2$. As $\partial_\xi^\alpha q_1.D_x^\alpha q_2 - c[\partial_\xi^\alpha q_{(m_1)}\wedge D_x^\alpha f_{(m_2)}]$ has Getzler order $\leq m_1+m_2-|\alpha|-1$ it follows that, for N large enough,

$$q_1\#q_2 = \sum_{|\alpha|\leq N}\frac{1}{\alpha!}c(\partial_\xi^\alpha q_{m_1}\wedge D_x^\alpha q_{m_2}) + \mathrm{O}_G(m_1+m_2-1). \tag{8.38}$$

Observe that $\sum\frac{1}{\alpha!}\partial_\xi^\alpha q_{(m_1)}\wedge D_x^\alpha q_{(m_2)}$ is exactly the symbol of $Q_{(m_1)}Q_{(m_2)}$ since $q_{(m_2)}(x,\xi,\tau)$ is polynomial in x and thus the sum is finite. Therefore, taking N large enough in (8.38) shows that the symbols of $\sigma[Q_1Q_2]$ and $Q_{(m_1)}Q_{(m_2)}$ coincide modulo a symbol of Getzler order $\leq m_1+m_2-1$. This proves the lemma. $\square$

Recall that by the Lichnerowicz formula,

$$D_E^2 = (\nabla^E)^*\nabla^E + \mathcal{F}^{E/\$} + \frac{1}{4}\kappa_M,$$

where

$$\mathcal{F}^{E/\$} = \frac{1}{2}\sum F^{E/\mathcal{S}}(e_i,e_j)c(e^i)c(e^j) = \frac{1}{2}\sum F^W(e_i,e_j)c(e^i)c(e^j).$$

Moreover, as explained in [**BGV**, p. 66],

$$(\nabla^E)^*\nabla^E = -g^{ij}(\nabla_i^E\nabla_j^E - \Gamma_{ij}^{\ k}\nabla_k),$$

where the g^{ij} are the coefficients of the inverse metric g^{-1} and the Γ_{ij}^k are the Christoffel symbols of g so that $\nabla_i^{TM}\partial_k = \Gamma_{ik}^{\ l}\partial_l$.

Combining this with Lemma 8.12.18 and the equations (8.24), (8.25) and (8.32) shows that D_E^2 has Getzler order 2 and model operator,

$$\begin{aligned} D_{E(2)}^2 &= -\delta_{ij}\nabla_{i(1)}\nabla_{j(1)} + \frac{1}{2}F^W(\partial_i,\partial_j)(0)dx^i\wedge dx^j \\ &= H_R + F^W(0), \qquad H_R := -\sum_{i=1}^n\left(\partial_i - \frac{1}{4}R_{ij}^M(0)x^j\right)^2. \end{aligned} \tag{8.39}$$

LEMMA 8.12.19. *Let Q be a Volterra parametrix for $D_E^2+\partial_t$. Then*

(1) *Q has Getzler order -2 and its model operator is $(H_R+F^W(0)+\partial_t)^{-1}$.*

(2) *We have*

$$K_{(H_R+F^W(0)+\partial_t)^{-1}}(0,x,t) = G_R(x,t)\wedge e^{-tF^W(0)}, \tag{8.40}$$

where $G_R(x,t)$ is the fundamental solution of $H_R+\partial_t$, i.e., the solution of the equation, $(H_R+F^W(0)+\partial_t)G_R(x,t) = \delta_(x,t)$, where $\delta(x,t)$ is the Dirac function on $\mathbb{R}^n\times\mathbb{R}$.

(3) *As $t\to 0^+$,*

$$\sigma[K_Q(0,0,t)]^{(2j)} = t^{j-\frac{n}{2}}[G_R(0,1)\wedge e^{-F^W(0)}]^{(2j)} + \mathrm{O}(t^{j-\frac{n}{2}+1}). \tag{8.41}$$

PROOF. Note that (3) follows by combining (1) and (2) with Lemma 8.12.17. Therefore, we only need to prove the first two assertions.

Let $p(x,\xi) = \sum p_j(x,\xi)$ be the symbol of $\not{D}^2$ and let $q\sim\sum q_{-2-j}$ be the symbol of Q. As $\not{D}^2$ is elliptic and has Getzler order 2 we have $p_{(2)}(0,\xi)^{(0)} = p_2(0,\xi)\neq 0$.

Hence $q_{-2} = (p_2 + i\tau)^{-1}$ has Getzler order -2. It then follows from (8.21) that each symbol q_{-2-j} has Getzler order ≤ -2, and hence Q has Getzler order -2.

Notice also that, as $(\not{D}^2 + \partial_t)Q - 1$ is smoothing, it follows from Lemma 8.12.18 the operator $(H_R + F^W(0) + \partial_t)Q_{(-2)} - 1$ has Getzler order ≤ -1. As it is Getzler-homogeneous of degree 0 it must be zero, and hence $Q_{(-2)} = (H_R + F^W(0) + \partial_t)^{-1}$. It is not difficult to check that, at the level of Schwartz kernels, the equality $Q_{(-2)}(H_R + F^W(0) + \partial_t) = 1$ means that

$$(H_{R,y} + F^W(0) + \partial_s)^T \left[K_{Q_{(-2)}}(x, y, t - s)\right] = \delta(x - y, t - s),$$

where $(H_{R,y} + F^W(0) + \partial_s)^T = H_{R,y} + F^W(0) - \partial_s$ is the transpose of $H_{R,y} + F^W(0) + \partial_s$. This implies that

$$(H_{R,y} + F^W(0) + \partial_s) \left[K_{Q_{(-2)}}(0, y, s)\right] = \delta(y, s),$$

that is, $K_{Q_{(-2)}}(0, x, t)$ is the fundamental solution of $H_R + F^W(0) + \partial_t$.

Observe that if we denote by $G_R(x, t)$ be the fundamental solution of $H_R + \partial_t$, then $G_{R,F}(x, t) := G_R(x, t) \wedge e^{-tF^W(0)}$ too is the fundamental solution of $H_R + F^W(0) + \partial_t$. Thus $K_{Q_{(-2)}}(0, x, t) = G_R(x, t) \wedge e^{-tF^W(0)}$. The proof is complete. □

At this stage observe that H_R is the harmonic oscillator associated to the skew-symmetric matrix $R^M(0) = (R^M_{ij}(0))$. We shall now make use of a version of the Melher's formula to determine the fundamental solution of $H_R + \partial_t$ (compare [**BGV**]).

Lemma 8.12.20 (Melher Formula). *Let $a > 0$ and consider the harmonic oscillator $H_a := -\frac{d}{dx^2} + \frac{1}{4}a^2x^2$ on $\mathbb{R}$. Then the fundamental solution of $H_a + \partial_t$ is*

$$G_a(x, t) := \chi(t)(4\pi t)^{-\frac{1}{2}} \left(\frac{at}{\sinh at}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{4t}x^2 \frac{at}{\tanh at}\right),$$

where $\chi(t)$ is the characteristic function of the interval $(0, \infty)$.

Proof. For $(x, t) \in \mathbb{R} \times (0, \infty)$ define

$$S_a(x, t) := (4\pi t)^{-\frac{1}{2}} \left(\frac{at}{\sinh at}\right)^{\frac{1}{2}} \exp(-\frac{1}{4t}x^2 \frac{at}{\tanh at}), \quad t > 0. \tag{8.42}$$

Observe that $S_a(x, t)$ is integrable on all products $\mathbb{R} \times (0, c)$ with $c > 0$ and

$$(H_a + \partial_t)S_a(x, t) = 0 \qquad \forall (x, t) \in \mathbb{R} \times (0, \infty).$$

Notice further that, as $t \to 0^+$,

$$\hat{S}_{x\to\xi}(\xi, t) = \cosh^{-\frac{1}{2}}(at) \exp(-\xi^2 t \, \frac{\tanh at}{at}) \longrightarrow 1,$$

uniformly on compact sets of $\mathbb{R}$. Thus, as $t \to 0^+$,

$$S(x, t) \longrightarrow \delta(x) \qquad \text{in } \mathcal{S}'(\mathbb{R}).$$

It follows from this that, for all u and v in $C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle (H_a+\partial_t)G_a(x,t), u(x)v(t)\rangle &= \int_0^\infty \int_{-\infty}^\infty S_a(x,t)(H_a-\partial_t)\left(u(x)v(t)\right)dxdt\\ &= \int_0^\infty \int_{-\infty}^\infty \left((H_a+\partial_t)S_a(x,t)\right)u(x)v(t)dxdt\\ &\qquad + \lim_{t\to 0^+}\left\{v(t)\int_{-\infty}^\infty S_a(x,t)u(x)dx\right\} = v(0)u(0).\end{aligned}$$

This shows that $(H_a+\partial_t)G_a(x,t)=\delta(x,t)$, that is, $G_a(x,t)$ is the fundamental solution of $H_a+\partial_t$. The lemma is thus proved. □

Lemma 8.12.21. *The fundamental solution $H_R+\partial_t$ is given by*

$$G_R(x,t)=\chi(t)(4\pi t)^{-\frac{n}{2}}\det{}^{\frac{1}{2}}\left(\frac{tR^M(0)/2}{\sinh(tR^M(0)/2)}\right)\exp\left(-\frac{1}{4t}\langle\frac{tR^M(0)/2}{\tanh(tR^M(0)/2)}x,x\rangle\right).$$

Proof. Let $A\in M_n(\mathbb{R})$ be a skew-symmetric matrix and set $B=-A^2$. Then an elaboration of Lemma 8.12.20 shows that the fundamental solution of the heat operator $-\sum_j \partial_j^2+\frac{1}{4}\langle Bx,x\rangle+\partial_t$ on $\mathbb{R}^n\times\mathbb{R}$ is given by

$$G_A(x,t)=\chi(t)(4\pi t)^{-\frac{n}{2}}\det{}^{\frac{1}{2}}\left(\frac{iAt}{\sinh(iAt)}\right)\exp\left(-\frac{1}{4t}\langle\frac{iAt}{\tanh(iAt)}x,x\rangle\right). \tag{8.43}$$

To see this we can notice that the $O(n)$-invariance of $G_A(x,t)$ enables us to reduce to the case where A is in normal form, i.e., it is a block-diagonal matrix of 2×2-matrices,

$$\begin{pmatrix} 0 & -a_j\\ a_j & 0\end{pmatrix}, \qquad a_j>0,$$

so that the eigenvalues of A are $\pm i a_j$ and B is a diagonal matrix with the a_j^2 as entries.

A further consequence of the $O(n)$-invariance of $G_A(x,t)$ is its invariance under rotations in the (x^j,x^k)-planes $j<k$. As the infinitesimal generator of the 1-parameter group of rotations in the (x^j,x^k)-plane is $x^j\partial_k-x^k\partial_j$, it follows that

$$(x^j\partial_k-x^k\partial_j)G_A(x,t)=0. \tag{8.44}$$

Consider the harmonic oscillator on $\mathbb{R}^n\times R$ associated to A, i.e.,

$$H_A:=-\sum_j\left(\partial_j-\sum_k\frac{i}{2}A_{jk}x^k\right)^2,$$

and observe that

$$\begin{aligned}H_A&=-\sum_j\partial_j^2-i\sum_{j,k}A_{jk}x^k\partial_j+\frac{1}{4}\sum_{jkl}A_{jk}A_{jl}x^kx^l\\ &=-\sum_j\partial_j^2+i\sum_{j<k}A_{jk}(x^j\partial_k-x^k\partial_j)+\frac{1}{4}\langle Bx,x\rangle.\end{aligned}$$

Therefore, in view of (8.44) we that $G_A(x,t)$ is also the fundamental solution of $H_A+\partial_t$. This means that, for all $u(x,t)\in C_c^\infty(\mathbb{R}^n\times\mathbb{R})$,

$$u(0,0)=\langle (H_A+\partial_t)G_A(x,t),u(x,t)\rangle\rangle=\langle G_A(x,t),(H_A-\partial_t)u(x,t)\rangle.$$

Observe that $\langle G_A(x,t), (H_A-\partial_t)u(x,t)\rangle$ is an analytic function of A. Therefore, substituting $-iR^M(0)/2$ for A in the formula (8.43) yields the fundamental solution of $H_R + \partial_t$. The proof is complete. $\square$

Combining Lemma 8.12.19 and Lemma 8.12.21 we get

$$\begin{aligned}\sigma\,[K_Q(0,0,t)]^{(n)} &= \left[G_R(0,1)\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t)\\ &= (4\pi)^{-\frac{n}{2}}\left[\det{}^{\frac{1}{2}}\left(\frac{R^M(0)/2}{\sinh(R^M(0)/2)}\right)\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t)\\ &= (4\pi)^{-\frac{n}{2}}\left[\hat{A}(R^M(0))\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t).\end{aligned}$$

Combining this with (8.28) and observing that $F^W(0) = F^{E/\$}(0)$, we deduce that

$$\mathrm{Str}\, k_t(0,0) = (2i\pi)^{-\frac{n}{2}}[\hat{A}(R^M(0))\wedge \mathrm{Ch}(F^{E/\$}(0))]^{(n)} + \mathrm{O}(t) \quad \text{as } t\to 0^+. \tag{8.45}$$

This proves (8.23) and completes the proofs of the local index theorem and local index formula of Atiyah-Singer.

Bibliography

[ABP] Atiyah, M., Bott, R., Patodi, V.: *On the heat equation and the index theorem.* Invent. Math. **19**, 279–330 (1973).

[AS] Atiyah, M., Singer, I.: *The index of elliptic operators. I, III* Ann. of Math. **87**, 484–530, 546–604 (1968).

[BGS] Beals, R., Greiner, P., Stanton, N.: *The heat equation on a CR manifold.* J. Differential Geom. **20**, 343–387 (1984).

[BGV] Berline, N., Getzler, E., Vergne, M.: *Heat kernels and Dirac operators.* Springer-Verlag, Berlin, 1992.

[Bi] Bismut, J.-M.: *The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem.* J. Funct. Anal. **57**, 56–99 (1984) .

[Ge1] Getzler, E.: *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem.* Comm. Math. Phys. **92**, 163–178 (1983).

[Ge2] Getzler, E.: *A short proof of the local Atiyah-Singer index theorem.* Topology **25**, 111–117 (1986).

[Gi] Gilkey, P.: *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Publish or Perish, 1984.

[Gr] Greiner, P.: *An asymptotic expansion for the heat equation.* Arch. Rational Mech. Anal. **41**, 163–218 (1971).

[Pi] Piriou, A.: *Une classe d'opérateurs pseudo-différentiels du type de Volterra.* Ann. Inst. Fourier **20**, 77–94 (1970).

[Po] Ponge, R.: *A new short proof of the local index formula and some of its applications.* Comm. Math. Phys. **241** (2003) 215–234.

[Sh] Shubin, M.: *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics. Springer-Verlag, 1987.

Appendix: Chern-Weil Construction of Characteristic Classes

$M^n = C^\infty$- manifold

E = real or complex vector bundle over M

∇^E = connection on E w/ curvature $F^E \in C^\infty(M, (\Lambda^2 T^* M) \otimes \mathrm{End}(E))$.

Define $\Omega^j(\mathrm{End}(E)) := C^\infty(M, \Lambda^j_{\mathbb{C}} T^* M \otimes \mathrm{End}(E))$.

$$(F^E)^j \in \Omega^{2j}(\mathrm{End}(E)) \leadsto (F^E)^j = 0 \text{ for } 2j > n.$$

Let $f(z) = \sum_{j\geq 0} a_j z^j$ be a power series in z. We define

$$f(F^E) := \sum_{j=0}^{\infty} a_j (F^E)^j$$

finite sum because $(F^E)^j = 0$ for $2j > n$.

$$= \sum_{0 \leq j \leq n/2} a_j (F^E)^j$$

$$\leadsto f(F^E) \in \Omega^{ev}(\mathrm{End}(E)) = \bigoplus_{j=0}^{[\frac{n}{2}]} \Omega^{2j}(\mathrm{End}(E)).$$

We have a fiberwise trace $\mathrm{Tr}_E : \mathrm{End}(E) \longrightarrow \underbrace{M \times \mathbb{C}}_{\text{trivial line bundle}}$.

This yields:

$$\mathrm{Tr}_E : \Lambda^*_{\mathbb{C}} T^* M \otimes \mathrm{End}(E) \longrightarrow \Lambda^*_{\mathbb{C}} T^* M$$
$$\eta \otimes A \longrightarrow \mathrm{Tr}_E(A)\eta$$

Def: $\mathrm{Tr}_E[f(F^E)] \in C^\infty(M, \Lambda^{ev}_{\mathbb{C}} T^* M)$ is the Chern-Weil form of F^E associated to $f(z)$.

Prop.: (1) $\mathrm{Tr}_E[f(F^E)]$ is a closed form.

(2) The cohomology class $[\mathrm{Tr}_E[f(F^E)]] \in H^{ev}(M, \mathbb{C})$ does not depend on the choice of the connection ∇^E on E.

Def.: The class $[\mathrm{Tr}[f(F^E)]]$ is called the characteristic class of E associated to $f(z)$.

Suppose now that $f(0) = 0$, i.e., $a_0 = 0$

$$\mathrm{Tr}_E[f(F^E)] = a_1 \mathrm{Tr}[F^E] + a_2 \mathrm{Tr}[(F^E)^2] + \ldots$$
$$= a_1 \mathrm{Tr}[F^E] \mod C^\infty(M, \Lambda^4_{\mathbb{C}} T^* M).$$

Thus,
$$\left[\mathrm{Tr}_E\, f(F^E)\right]^{j} = a_i^{j}\left[\mathrm{Tr}_E(F^E)\right]^{j} \mod C^\infty(M, \Lambda^{2j+2}T^*M)$$
$$= 0 \quad \text{if } 2j > n.$$
Therefore, we can define
$$\exp\left[\mathrm{Tr}_E\, f(F^E)\right] := \sum_{j=0}^{\infty} \frac{1}{j!}\left[\mathrm{Tr}_E\, f(F^E)\right]^{j}$$
$$= \sum_{0\leq j<\frac{n}{2}} \frac{1}{j!}\left[\mathrm{Tr}_E\, f(F^E)\right]^{j} \quad \text{(finite sum)}$$

Using the previous proposition it is not difficult to check that each form $\left[\mathrm{Tr}_E\, f(F^E)\right]^{j}$ is closed and its cohomology

Prop.: The form $\exp\left[\mathrm{Tr}_E\, f(F^E)\right]$ is closed and its cohomology class in $H^{ev}(M,\mathbb{C})$ does not depend on the choice of the connection ∇^E on E.

Recall that if A is a positive definite $n\times n$ matrix, then
$$\det A = \exp\left\{\mathrm{Tr}(\log A)\right\}.$$

Let $g(z) = \sum_{\ell=0}^{\infty} b_\ell z^\ell$ be a power series s.t. $g(0)=1$. Then $\log g(z) = \log(1+(g(z)-1))$ is a well defined power series. Therefore, for any $\alpha\in\mathbb{R}_+^*$, we can define
$$\det{}^\alpha g(z) := \exp\left\{\alpha\, \mathrm{Tr}_E\left[\log(g(F^E))\right]\right\}$$

Example 1: Chern Character:

Let E be a complex vector bundle.

Def.: (1) The (total) Chern form of F^E is
$$\mathrm{Ch}(F^E) := \mathrm{Tr}_E\left(e^{-F^E}\right)$$
(2) The characteristic class $[\mathrm{Ch}(F^E)]$ is denoted $\mathrm{Ch}(E)$ and called the Chern character of E.

Prop.: Let E_1, E_2 be complex vector bundles over M. Then

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2),$$
$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \wedge \mathrm{ch}(E_2).$$

Example 2: The $\hat{A}$-Genus:
Let E be a real vector bundle over M.

Def.: (1) The $\hat{A}$-form of F^E is

$$A(F^E) := \det{}^{1/2}\left[\frac{F^E/2}{\sinh(F^E/2)}\right].$$

(2) The characteristic class $[A(F^E)]$ is called the $\hat{A}$-genus of E and is denoted $\hat{A}(E)$.
(3) If $E = TM$, then $\hat{A}(TM)$ is simply denoted $\hat{A}(M)$ and called the $\hat{A}$-genus of M.

Prop. Let E_1 and E_2 be real vector bundles over M. Then

$$\hat{A}(E_1 \oplus E_2) = \hat{A}(E_1) \wedge \hat{A}(E_2).$$

Example 3: The Todd Genus:
Let E be a complex vector bundle over M.

Def.: (1) The Todd form of F^E is

$$\mathrm{Td}(F^E) := \det\left[\frac{F^E}{e^{F^E}-1}\right].$$

(2) The characteristic class $[\mathrm{Td}(F^E)]$ is called the Todd genus of E and is denoted $\mathrm{Td}(E)$.
(3) If M is a complex manifold and $E = TM$, then $\mathrm{Td}(TM)$ is simply denoted $\mathrm{Td}(M)$ and is called the Todd genus of M.

Prop.: If E_1 and E_2 are complex vector bundles over M, then

$$\mathrm{Td}(E_1 \oplus E_2) = \mathrm{Td}(E_1) \wedge \mathrm{Td}(E_2).$$