CHAPTER I

Spectrum and Duality Between Spaces and Algebras

I.1. Banach algebras and C^*-algebras

In the sequel all the vector spaces and algebras are vector spaces or algebras over $\mathbb{C}$. Unless otherwise mentioned all the topological spaces are assumed to be Hausdorff.

DEFINITION I.1. *A Banach algebra is an algebra A endowed with a Banach norm $\|.\|$ such that*

$$\|xy\| \leq \|x\|\|y\| \qquad \forall x, y \in A.$$

DEFINITION I.2. *A C^*-algebra is a Banach algebra A together with an antilinear involution $x \to x^*$ such that*

$$(xy)^* = y^* x^* \qquad \forall x, y \in A, \tag{I.1.1}$$

$$\|x^*\| = \|x\| \quad \textit{and} \quad \|x^* x\| = \|x\|^2 \qquad \forall x \in A. \tag{I.1.2}$$

DEFINITION I.3. *Let A and B be C^*-algebras.*

(1) *A $*$-homomorphism $\phi : A \to B$ is a continuous homomorphism of algebras such that $\phi(x^*) = x^*$ for all $x \in A$.*
(2) *A $*$-isomorphism $\phi : A \to B$ is $*$-homomorphism which is bijective.*

Recall that by the open mapping theorem any bijective continuous linear map between Banach spaces has a continuous inverse, so the inverse of any $*$-isomorphism is continuous. As we will see later any $*$-isomorphism between C^*-algebras is isometric (cf. Proposition I.5).

EXAMPLE I.4. Let $M_n(\mathbb{C})$ be the algebra of $n \times n$-matrices with complex entries. It is equipped with the involution $A \to A^*$, where $A^* := \overline{A}^t$ is the adjoint of A. A C^*-algebra norm is given by the norm defined by

$$\|A\| := \sup\{\|Ax\|;\ x \in \mathbb{C}^n,\ |x| = 1\} \qquad \forall A \in M_n(\mathbb{C}).$$

EXAMPLE I.5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of continuous linear operators $T : \mathcal{H} \to \mathcal{H}$. The algebra $\mathcal{L}(\mathcal{H})$ comes equipped with the involution $T \to T^*$, where T^* is the adjoint of T, i.e., the unique linear operator on $\mathcal{H}$ such that

$$\langle T^* \xi, \eta \rangle = \langle \xi, \eta \rangle \quad \forall \xi, \eta \in \mathcal{H}.$$

The C^*-algebra norm of $\mathcal{L}(\mathcal{H})$ is given by

$$\|T\| := \sup_{\|\xi\|=1} \|T\xi\|$$

More generally, any closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$ is a C^*-algebra.

Definition I.6. *A $*$-representation of a C^*-algebra in a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from A to $\mathcal{L}(\mathcal{H})$.*

Theorem I.1 (Gel'fand-Naimark; see [**GVF**, pp. 29–31]). *Any C^*-algebra A admits an isometric $*$-representation π into some Hilbert space $\mathcal{H}$.*

Recall the following.

Lemma I.1. *Let $T : E_1 \to E_2$ be a linear map between Banach spaces which is isometric, i.e.,*

$$\|Tx\|_{E_2} = \|x\|_{E_1} \qquad \forall x \in E_1.$$

Then the image $T(E_1)$ is a closed subspace of E_2 and T gives rise to an isometric isomorphism from E_1 onto $T(E_1)$.

Proof. Exercise! □

Using this lemma we see that in Gel'fand-Naimark's theorem, $\pi(A)$ is a closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$, and hence is a C^*-algebra. Moreover, the representation π provides with an isometric $*$-isomorphism of A onto $\pi(A)$. Therefore, we see that any C^*-algebra can be represented as sub-C^*-algebra of some $\mathcal{L}(\mathcal{H})$.

Example I.7. Let X be a compact (Hausdorff) space and let $C(X)$ be its algebra of continuous complex-valued functions. The algebra has the natural involution $f \to \overline{f}$, where $\overline{f}$ denotes the complex conjugate of f. The C^*-algebra norm of $C(X)$ is the usual norm of $C(X)$, i.e.,

$$\|f\|_{C(X)} := \sup_{x \in X} |f(x)| \qquad \forall f \in C(X).$$

Notice that the constant function 1 is a unit for $C(X)$, so $C(X)$ is a unital commutative C^*-algebra.

Example I.8. Let X be a compact topological space and let $C_0(X)$ be its algebra of continuous functions "vanishing at infinity". Recall that a function $f \in C(X)$ vanishes at infinity if, for all $\epsilon > 0$, there exists a compact $K \subset X$ such that $|f(x)| \leq \epsilon$ for all $x \in X \setminus K$. As in the previous example $C_0(X)$ has the natural involution given by taking complex conjugates and its C^*-algebra is its usual norm,

$$\|f\|_{C_0(X)} := \sup_{x \in X} |f(x)| \qquad \forall f \in C_0(X).$$

The C^*-algebra $C_0(X)$ is commutative, but as it does not contain the constant function 1 this is not a unital C^*-algebra.

As we will see later in Section I.4 the previous two examples are essentially the only examples of commutative C^*-algebras.

Finally, there are some notions Banach algebras (e.g. spectrum; see below) that requires the Banach algebra to have a unit. However, there is a way to add a unit to any Banach algebra A. This can be explained as follows. Define

$$A^+ = A \oplus \mathbb{C}.$$

This is a vector space. We turn it into a Banach algebra by means of the product and norm defined by

$$(x_1, \lambda_1).(x_2, \lambda_2) := (x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1 \lambda_2) \qquad \forall (x_j, \lambda_j) \in A^+,$$
$$\|(x, \lambda)\|_{A^+} := \inf\{\|xy + \lambda y\|_A;\ y \in A,\ \|y\|_A = 1\} \qquad \forall (x, \lambda) \in A^+.$$

In particular A^+ has unit

$$1_{A^+} := (0, 1).$$

The map $x \to (x, 0)$ provides us with an isometric embedding of A into A^+. This allows us to identify A with the closed ideal $A \otimes \{0\}$ of A^+. Using this identification we can write any element of A^+ as

$$(x, \lambda) = x + \lambda 1_{A^+}, \qquad x \in A, \quad \lambda \in \mathbb{C}.$$

If A is C^*-algebra we further endow A^+ with the involution

$$(x, \lambda) \longrightarrow (x^*, \bar{\lambda}).$$

This turns A^+ into a C^*-algebra and then A embeds into A^+ as a C^*-algebra.

I.2. Spectrum

In this section we let A be a Banach algebra with a unit 1_A such that $\|1_A\| = 1$. This latter condition is always satisfied when A is a (unital) C^*-algebra (exercise!).

In the sequel we denote by A^{-1} the group of invertible elements of A. This is an open subset of A.

Definition I.9. *Let $x \in A$. The spectrum of x is*

$$\mathrm{Sp}_A(x) := \{\lambda \in \mathbb{C};\ x - \lambda \notin A^{-1}\}.$$

The complement of $\mathrm{Sp}_A(x)$ is called the resolvent set of x.

Remark I.10. When A is not unital, we define the spectrum of $x \in A$ to its spectrum in A^+, i.e., $\mathrm{Sp}_A(x) := \mathrm{Sp}_{A^+}(x)$. Notice that 0 is always contained in $\mathrm{Sp}_{A^+}(x)$ because as A is a non-trivial ideal of A^+ none of its elements can be invertible in A^+. In particular, when A is unital $\mathrm{Sp}_{A^+}(x) = \mathrm{Sp}_A(x) \cup \{0\}$.

Proposition I.1. *Let $x \in A$. Then*

(1) *The spectrum $\mathrm{Sp}_A(x)$ is a non-empty compact subset of $\mathbb{C}$.*
(2) *The resolvent $\lambda \to (x - \lambda)^{-1}$ is an analytic map from $\mathbb{C} \setminus \mathrm{Sp}_A(x)$ to A.*

Definition I.11. *Let $x \in A$. The spectrum radius of x is*

$$\rho(x) = \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}_A(x)\}.$$

Proposition I.2 (Gel'fand-Mazur; see [**Ar**, Thm. 1.7.3]). *For all $x \in A$,*

$$\rho(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} \leq \|x\|.$$

Proposition I.3. *Suppose that A is a C^*-algebra and let $x \in A$ be normal (i.e. $x^*x = xx^*$). Then*

$$\rho(x) = \|x\|.$$

Proof. As A is a C^*-algebra and $x^*x = xx^*$, we have

$$\|x^2\| = \|(x^2)^*x^2\|^{\frac{1}{2}} = \|(x^*x)^*(x^*x)\|^{\frac{1}{2}} = \|x^*x\| = \|x\|^2.$$

An induction then shows that

$$\|x^{2^n}\| = \|x\|^{2^n} \qquad \forall n \in \mathbb{N}.$$

Combining this with Proposition I.2 gives

$$\rho(x) = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|,$$

proving the proposition. □

In the sequel if A is a Banach algebra equipped with an antilinear, we called C^*-norm a Banach norm satisfying (I.1) and (I.1.2).

Proposition I.4. *If A is a C^*-algebra then its norm is its unique C^*-norm.*

Proof. Possibly by embedding A into A^+ we may assume that A is unital. Assume that $\|.\|'$ is another C^*-norms on A. The spectrum of an element $x \in A$ is independent of the choice of the norm. Therefore, if $x \in A$ is normal, then from Proposition I.3 we get

$$\|x\| = \rho(x) = \|x\|'.$$

In general, even if x is not normal, x^*x is. Therefore, using the fact that $\|.\|$ and $\|.\|'$ are C^*-norms we get

$$\|x\|^2 = \|x^*x\| = \|x^*x\| = \|x\|'^2.$$

Thus the norms $\|.\|$ and $\|.\|'$ agree, proving the proposition. $\square$

Proposition I.5. *Every $*$-isomorphism between C^*-algebras is isometric.*

Proof. If $\phi : A_1 \to A_2$ is a $*$-isomorphism between C^*-algebras A_1 and A_2, then $\|\phi(x)\|_{A_2}$ is a C^*-norm on A_1. Therefore, by Proposition I.4 it agrees with the original norm of A_1, i.e., ϕ is an isometry. $\square$

I.3. Holomorphic Functional Calculus

Let A be a Banach algebra with a unit 1_A such that $\|1_A\|$. Let $x \in A$ and set $S = \mathrm{Sp}_A(x)$. In addition, let Ω be an open subset of $\mathbb{C}$ containing S and denote by $\mathrm{Hol}(\Omega)$ the algebra of holomorphic functions on Ω. As usual we endow $\mathrm{Hol}(\Omega)$ with the topology of uniform convergence on compact subsets of Ω.

Let $f \in \mathrm{Hol}(\Omega)$. Then by Cauchy's formula, for all $z \in S$, we have

$$f(z) = \frac{1}{2i\pi}\int_\Gamma \frac{f(\lambda)}{\lambda - z} d\lambda,$$

where Γ is any continuous and piecewise-C^1 bounded contour contained in Ω such that $\mathrm{ind}_\Gamma(\lambda) = 1$ for all $\lambda \in S$.

Recall that the map $\lambda \to (\lambda - x)^{-1}$ is analytic from $\mathbb{C} \setminus S$ to A. Therefore, we can define

$$f(x) = \frac{1}{2i\pi}\int_\Gamma \frac{f(\lambda)}{\lambda - x} d\lambda, \tag{I.3.1}$$

where Γ is as above and the integral is defined as a limit of Riemann sums in A. This integral does not depend on the choice of Γ.

Theorem I.2 (Holomorphic Functional Calculus; see [**Yo**, pp. 226-228]). *Let Ω be an open containing $S = \mathrm{Sp}_A\, x$.*

(1) *The map $f \to f(x)$ is a continuous unital homomorphism of algebras from $\mathrm{Hol}(\Omega)$ to A.*
(2) *If f and g are elements of $\mathrm{Hol}(\Omega)$ that agree on S, then $f(x) = g(x)$.*
(3) *For all $f \in \mathrm{Hol}(\Omega)$ we have*

$$\mathrm{Sp}_A\, f(x) = f(S).$$

EXAMPLE I.12. Let $f(z) = \sum_{n=0}^{\infty}$ be a power series with convergence radius $R > \|x\|$, and let Γ be a (direct-oriented) circle about the origin with radius r, $\|x\| < r < R$. Then

$$f(x) = \frac{1}{2i\pi}\int_\Gamma \left(\sum_{n=0}^{\infty} a_n\lambda^n\right)(\lambda - x)^{-1}d\lambda = \sum_{n=0}^{\infty}\frac{a_n}{2i\pi}\int_\Gamma \lambda^n(\lambda - x)^{-1}d\lambda. \tag{I.3.2}$$

If $|\lambda| > \|x\|$ then $\|\lambda^{-1}x\| < 1$, and so we have

$$(\lambda - x)^{-1} = \lambda^{-1}(1 - \lambda^{-1}x) = \sum_{k=0}^{\infty}\lambda^{-(k+1)}x^n,$$

where the series converges normally. Therefore, for $n = 0, 1, \ldots$ we have

$$\frac{1}{2i\pi}\int_\Gamma \lambda^n(\lambda - x)^{-1}d\lambda = \sum_{k=0}^{\infty}\left(\frac{1}{2i\pi}\int_\Gamma \lambda^{n-(k+1)}d\lambda\right)x^n = x^n. \tag{I.3.3}$$

Combining this with (I.3.2) shows that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Notice that for $n = 0$ Eq. (I.3.3) shows that when $f(z)$ is the constant function 1 we have $f(x) = 1$, that is, Φ is a unital homomorphism of algebras.

This result has a couple of important applications.

PROPOSITION I.6. *Let B be a closed subalgebra of A containing the unit 1_A.*
(1) *If $x \in B$ is invertible in A, then x^{-1} belongs to B.*
(2) *For all $x \in B$ we have*

$$\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x).$$

PROOF. Let $x \in B$ be invertible in A. By Proposition I.2 both $\mathrm{Sp}_A(x)$ and $\mathrm{Sp}_B(x)$ are contained in the disk $\overline{D(0, \|x\|)}$. Let Γ be a (direct-oriented) circle about the origin with radius $r > \|x\|$. By assumption 0 is not contained in $\mathrm{Sp}_A(x)$, so $i(z) = z^{-1}$ is a holomorphic function near $\mathrm{Sp}_A(x)$. Therefore the holomorphic functional calculus allows to define $i(x)$ as an element of A. Since the holomorphic functional calculus for x is a homomorphism of algebras from $\mathrm{Hol}(\mathbb{C}\setminus 0)$ to x and $z.i(z) = 1$ on $\mathbb{C}\setminus 0$, we see that $xi(x) = 1$, that is, $i(x) = x^{-1}$. Thus,

$$x^{-1} = i(x) = \frac{1}{2i\pi}\int_\Gamma \frac{\lambda^{-1}}{\lambda - x}d\lambda.$$

As mentioned above $\mathrm{Sp}_B(x)$ too is contained in $\overline{D(0, \|x\|)}$, so $\lambda - x$ is invertible in B for all $\lambda \in \mathbb{C}\setminus\overline{D(0, \|x\|)}$. Thus $\lambda^{-1}(\lambda - x)^{-1}$ is an element of B for all $\lambda \in \Gamma$. Since B is a closed subspace of A it follows that the integral in (I.3) converges in B, proving that x^{-1} is contained in B.

It follows from this that $B^{-1} = A^{-1}\cap B$. Therefore, if $x \in B$ then, for all $\lambda \in \mathbb{C}$, we have

$$x - \lambda \in B^{-1} \Longleftrightarrow x - \lambda \in A^{-1}\cap B \Longleftrightarrow x - \lambda \in A^{-1},$$

proving that $\mathrm{Sp}_B(x)$ and $\mathrm{Sp}_A(x)$ agree. The proof is complete. □

In the sequel we denote by S^1 the unit sphere of $\mathbb{C}$.

Proposition I.7. *Assume that A is a C^*-algebra.*

(1) *If $x \in A$ is unitary (i.e. $x^*x = xx^* = 1$), then* $\operatorname{Sp}_A(x) \subset S^1$.
(2) *If $x \in A$ is selfadjoint (i.e. $x^* = x$), then* $\operatorname{Sp}_A(x) \subset \mathbb{R}$.

Proof. Let $x \in A$ be unitary. As $\|x\|^2 = \|x^*x\| = \|1_A\| = 1$, using Proposition I.3 we see that $\operatorname{Sp}_A(x)$ is contained in the closed unit disk $\overline{D(0,1)}$. Similarly $\operatorname{Sp}_A x^*$ is contained in $\overline{D(0,1)}$ too. By assumption $x^* = x^{-1}$, so using Theorem I.2 we deduce that $\operatorname{Sp}_A x^* = \operatorname{Sp}_A(x^{-1}) = (\operatorname{Sp}_A x)^{-1}$. Therefore $(\operatorname{Sp}_A x)^{-1}$ is contained in $\overline{D(0,1)}$, and hence $\operatorname{Sp}_A x$ is contained in $\mathbb{C} \setminus D(0,1)$. Thus $\operatorname{Sp}_A x$ is contained in $\overline{D(0,1)} \cap (\mathbb{C} \setminus D(0,1)) = S^1$.

Let $x \in A$ be selfadjoint. Set $u = \exp(ix)$. Here $\exp(ix)$ is defined by holomorphic functional calculus but, as shown in Example I.12,

$$\exp(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}. \tag{I.3.4}$$

Thus,

$$u^* = \sum_{n=0}^{\infty} \frac{((ix)^n)^*}{n!} = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} = \exp(-ix).$$

As the holomorphic functional calculus for x is an algebra homomorphism and $\exp(-iz)\exp(iz) = 1$ for all $z \in \mathbb{C}$, we deduce that $u^*u = \exp(-ix)\exp(ix) = 1$. Similarly $uu^* = 1$, so u is unitary. It then follows from the first part that $\operatorname{Sp}_A u$ is contained in S^1.

By Theorem I.2 we know that $\operatorname{Sp}_A(u) = \operatorname{Sp}_A \exp(ix) = \exp(i \operatorname{Sp}_A x)$, so we see that $\exp(i\lambda)$ is contained in S^1 for all $\lambda \in \operatorname{Sp}_A x$. This implies that $\operatorname{Sp}_A x$ is contained in $\mathbb{R}$. The proof is complete. □

I.4. The Gel'fand Transform

In this section we let A be a unital commutative C^*-algebra.

Definition I.13. *A character of A is a linear map $\chi : A \to \mathbb{C}$ such that*

$$\chi(xy) = \chi(x)\chi(y) \qquad \forall x, y \in A, \tag{I.4.1}$$

$$\chi(1_A) = 1. \tag{I.4.2}$$

The set of characters of A is called the Gel'fand spectrum of A and is denoted $\operatorname{Sp} A$.

It can be shown that the characters of A are in one-to-one correspondence with maximal ideals of A (see [**GVF**, p. 7]).

Example I.14. Let X be a compact space and consider the C^*-algebra $C(X)$ of continuous functions X. For any $x \in X$ the evaluation by x defines a linear form on $C(X)$, namely,

$$\chi_x(f) := f(x) \qquad \forall f \in C(X). \tag{I.4.3}$$

Then χ_x is a character of $C(X)$. Furthermore, it can be shown (cf. [**GVF**, pp. 9-10]) that

$$\operatorname{Sp} C(X) = \{\chi_x;\ x \in X\}. \tag{I.4.4}$$

Thus the characters of $C(X)$ are in one-to-one correspondence with the points of X.

The connection between the spectrum of A and the spectra of its points is given by the following.

Proposition I.8. *For all $x \in A$ we have*

$$\mathrm{Sp}_A(x) = \{\chi(x);\ \chi \in \mathrm{Sp}\, A\}.$$

Proof. For all $\lambda \in \mathbb{C} \setminus \mathrm{Sp}_A(x)$ and all $\chi \in \mathrm{Sp}\, A$ we have

$$\chi\left((x-\lambda)^{-1}\right)(\chi(x)-\lambda) = \chi\left((x-\lambda)^{-1}(x-\lambda)\right) = \chi(1) = 1,$$

and hence $\chi(x) \neq \lambda$. This shows that $\mathbb{C} \setminus \mathrm{Sp}_A(x)$ is contained in the complement of $S := \{\chi(x);\ \chi \in \mathrm{Sp}\, A\}$, that is, S is contained in $\mathrm{Sp}_A(x)$. The converse inclusion is the content of Lemma 1.3 of [**GVF**]. $\square$

This result has a few important consequences.

Proposition I.9. *Let $\chi \in \mathrm{Sp}\, A$. Then*

$$\chi(x^*) = \overline{\chi(x)} \qquad \forall x \in A.$$

Proof. Let $x \in A$. Assume first that x is selfadjoint. By Proposition I.8 we know that $\chi(x)$ is contained in $\mathrm{Sp}_A(x)$. Since x is selfadjoint Proposition I.7 tells us that $\mathrm{Sp}_A\, x$ is contained in $\mathbb{R}$. Therefore we see that $\chi(x)$ is real number.

In general, set $x_1 = \frac{1}{2}(x + x^*)$ and $x_2 = \frac{1}{2i}(x - x^*)$. Then x_1 and x_2 both are selfadjoint and $x = x_1 + ix_2$. By linearity,

$$\chi(x) = \chi(x_1 + ix_2) = \chi(x_1) + i\chi(x_2).$$

Similarly,

$$\chi(x^*) = \chi(x_1 - ix_2) = \chi(x_1) - i\chi(x_2).$$

As x_1 and x_2 are selfadjoint both $\chi(x_1)$ and $\chi(x_2)$ are real numbers. Therefore $\chi(x_1) - i\chi(x_2) = \overline{\chi(x_1) + i\chi(x_2)}$, that is, $\chi(x^*) = \overline{\chi(x)}$, proving the proposition. $\square$

In the sequel we denote by A^* the topological dual of A. This is a Banach space equipped with the norm,

$$\|\varphi\|_{A^*} := \sup_{\|x\|=1} |\langle \varphi, x\rangle| \qquad \forall \varphi \in A^*.$$

We denote by Ω the unit sphere of A^*.

The weak-$*$ topology on A^* is given by the topology of pointwise convergence. By the Banach-Alaoglu theorem Ω is compact with respect to the weak-$*$ topology.

Proposition I.10. $\mathrm{Sp}\, A$ *is contained in* Ω.

Proof. Let $\chi \in \mathrm{Sp}\, A$ and let $x \in A$. Thanks to Proposition I.8 we know that $\chi(x)$ is contained in $\mathrm{Sp}_A(x)$, and so using Proposition I.3 we get

$$|\chi(x)| \leq \rho(x) \leq \|x\|.$$

This shows that χ is a continuous linear form on A and we have $\|\chi\|_{A^*} \leq 1$. Moreover, as $\chi(1_A) = 1$ and $\|1_A\| = 1$ (since A is a C^*-algebra), we see that $\|\chi\|_{A^*} = 1$, that is, χ belongs to Ω, proving the proposition. $\square$

Notice that the maps $\chi \to \chi(xy) - \chi(x)\chi(y)$, $x, y \in A$, and $\chi \to \chi(1)$ are continuous on A^* with respect to the weak-$*$ topology. Therefore the conditions (I.4.1)–(I.4.2) defining $\mathrm{Sp}\, A$ imply that $\mathrm{Sp}\, A$ is a closed subset of A^* with respect to weak-$*$ topology. Since by the previous proposition $\mathrm{Sp}\, A$ is contained Ω, which is compact with respect to the weak-$*$ topology, we obtain:

Proposition I.11. *Equipped with the weak-$*$ topology* $\mathrm{Sp}\, A$ *is a compact space.*

In particular we can consider the C^*-algebra $C(\operatorname{Sp} A)$. This a unital commutative C^*-algebra.

Definition I.15. *The Gel'fand transform of A is the map $G : A \to C(\operatorname{Sp} A)$ defined by*

$$G(x) = \hat{x}, \qquad \hat{x}(\chi) := \chi(x) \qquad \forall x \in A\ \forall \chi \in \operatorname{Sp} A.$$

It is immediate to check that G is a homomorphism of algebras.

Theorem I.3 (Gel'fand-Naimark). *The Gel'fand transform is an (isometric) $*$-isomorphism from A onto $C(\operatorname{Sp} A)$.*

Proof. Let $x \in A$. Using Proposition I.9 we see that, for all $\chi \in \operatorname{Sp} A$,

$$G(x^*)(\chi) = \chi(x^*) = \overline{\chi(x)} = \overline{G(x)(\chi)},$$

that is, $G(x^*) = \overline{G(x)}$, proving that G is a $*$-homomorphism.

By Proposition I.8 we have

$$\operatorname{Sp}_A(x) = \{\chi(x);\ \chi \in \operatorname{Sp} A\} = \{G(x)(\chi);\ \chi \in \operatorname{Sp} A\} = \operatorname{Sp}_{C(\operatorname{Sp} A)} G(x),$$

and hence $\rho(x) = \rho(G(x))$. In particular, if x is normal, then so is $G(x)$ and using Proposition I.3 we get

$$\|G(x)\| = \rho(G(x)) = \rho(x) = \|x\|.$$

In general, as A is a C^*-algebra we have

$$\|x\|^2 = \|x^* x\| = \|G(x^* x)\| = \|\overline{G(x)}G(x)\| = \|G(x)\|^2,$$

proving that G is an isometry. It then follows from Lemma I.1 that $G(A)$ is a closed subalgebra $C(\operatorname{Sp} A)$ and G gives rise to an isometric isomorphism from A onto $G(A)$.

To complete the proof we need to show that G is onto. As we have just seen $G(A)$ is sub-C^*-algebra of $C(\operatorname{Sp} A)$. It contains the constant function $1 = G(1_A)$. Moreover, if χ_1 and χ_2 are two distinct elements of $\operatorname{Sp} A$, then there exists $x \in A$ such that $\chi_1(x) \neq \chi_2(x)$, that is, $G(x)(\chi_1) \neq G(x)(\chi_2)$. Thus $G(A)$ separates the points of $\operatorname{Sp} A$. We then can apply the Stone-Weierstrass theorem (as in its version in [**Fo**, 4.51]) to conclude that $G(A) = C(\operatorname{Sp} A)$, completing the proof. ☐

Combining this with Example I.14 shows that the Gel'fand transform provides us with an equivalence,

$$\begin{array}{ccc} \{\text{Compact Hausdorff spaces}\} & & \{\text{Unital commutative } C^*\text{-algebras}\} \\ X & \longrightarrow & C(X) \\ \operatorname{Sp} A & \longleftarrow & A \end{array}$$

This actually gives rise to an equivalence of categories (cf. [**GVF**, pp. 9–16]). Thus general unital C^*-algebras can be seen as the noncommutative analogue of compact spaces.

When A is not unital we defined its characters as the non-zero homomorphisms $\chi :\to \mathbb{C}$ (when A is unital any non-zero homomorphism $\chi : A \to \mathbb{C}$ is unital; exercise!). Denote by $\operatorname{Sp} A$ the Gel'fand spectrum of A, that is, its set of characters. Then $\operatorname{Sp} A$ is contained in the (closed) unit ball $B_{A^*}(0,1)$ of A^* and it can be shown to be a locally compact space. Indeed, consider

$$K = \{\chi \in B_{A^*}(0,1); \chi(xy) = \chi(x)\chi(y) \forall x, y \in A\}.$$

Then $\operatorname{Sp} A = K \setminus \{0\}$. Noticing that K is a closed subset of $B_{A^*}(0,1)$, and hence is compact with respect to the weak-* topology of A^*, we see that $\operatorname{Sp} A$ becomes a locally compact once endowed with the weak-* topology of A^*.

The Gel'fand transform $G : A \to C(\operatorname{Sp} A)$ is defined as in Definition I.15. It can be shown to give rise to an isometric $*$-isomorphism $G : A \to C_0(\operatorname{Sp} A)$. Therefore, in this case again, the Gel'fand transform provided us with an equivalence,

$$\begin{array}{ccc} \{\text{Locally compact Hausdorff spaces}\} & & \{\text{Commutative } C^*\text{-algebras}\} \\ X & \longrightarrow & C_0(X) \\ \operatorname{Sp} A & \longleftarrow & A \end{array}$$

I.5. Continuous Functional Calculus

In this section we let A be a C^*-algebra and we let $x \in A$ be normal (i.e., $x^*x = xx^*$). Possibly by embedding A into A^+ we may assume that A is unital. In addition we denote by S the spectrum of x.

Let $\mathcal{P}$ be the space of polynomials $f(z) = \sum_{m,} c_{mn} z^m \bar{z}^n$ in the variables z and $\bar{z}$. We shall regard $\mathcal{P}$ as an involutive subalgebra of $C(S)$. If $f = \sum_{n,m} c_{mn} z^n \bar{z}^m$ is an element of $\mathcal{P}$ we set

$$f(x) = \sum_{m,n} c_{mn} x^m (x^*)^m.$$

The map $f \to f(x)$ is a unital $*$-homomorphism of algebras from $\mathcal{P}$ to A. Let us denote by $\mathcal{B}$ its image. This is an involutive unital subalgebra of A, so its closure $B := \overline{\mathcal{B}}$ is a unital C^*-algebra. The latter is the C^*-algebra generated by x. As x is normal this a commutative C^*-algebra.

Since B is a unital commutative C^*-algebra, the Gel'fand transform $G : B \to C(\operatorname{Sp} B)$ is an isometric unital isomorphism of C^*-algebras. Set $a = G(x)$; this is an element of $C(\operatorname{Sp} B)$. If $f = \sum_{n,m} c_{mn} z^n \bar{z}^m$ is an element of $\mathcal{P}$, then

$$G(f(x)) = G\left(\sum_{m,n} c_{mn} x^m (x^*)^n\right) = \sum_{m,n} c_{mn} G(x)^m \overline{G(x)}^n = f \circ a.$$

Thus,

$$f(x) = G^{-1}(f \circ a) \qquad \forall f \in \mathcal{P}. \tag{I.5.1}$$

Since G is an isomorphism $a(\operatorname{Sp} B) = \operatorname{Sp}_{C(\operatorname{Sp} B} G(x) = \operatorname{Sp}_B x$, so using Proposition I.6 we see that $a(\operatorname{Sp} B) = \operatorname{Sp}_A x = S$. This means that for all $f \in C(S)$ the composite $f \circ a$ is a well-defined element of $C(\operatorname{Sp} B)$.

Definition I.16. *For all $f \in C(S)$ we define*

$$f(x) := G^{-1}(f \circ a) \in A.$$

Theorem I.4 (Continuous Functional Calculus). *The following holds.*

1. *The map $\Phi : x \to f(x)$ is an isometric $*$-homomorphism from $C(S)$ to A, whose image is the C^*-algebra generated by x.*
2. *If $f = \sum_{n,m} c_{mn} z^n \bar{z}^m$ is an element of $\mathcal{P}$, then*

$$f(x) = \sum_{m,n} c_{mn} x^m (x^*)^n. \tag{I.5.2}$$

(3) *For all $f \in C(S)$ we have*

$$\operatorname{Sp}_A f(x) = f(S). \tag{I.5.3}$$

PROOF. First, (I.5.2) immediately follows from (I.5.1). In particular we see that the algebra $\mathcal{B}$ is contained in the image of Φ.

Observe that the map $\Psi : f \to f \circ a$ is a unital $*$-homomorphism from $C(S)$ to $C(\operatorname{Sp} B)$. In addition, using the fact that $a(\operatorname{Sp} B) = S$ we see that, for all $f \in C(S)$,

$$\|f \circ a\|_{C(\operatorname{Sp} B)} = \sup_{\chi \in \operatorname{Sp} B} |f(a(\chi))| = \sup_{z \in a(\operatorname{Sp} B)} |f(z)| = \sup_{z \in S} |f(z)| = \|f\|_{C(S)}.$$

Therefore Ψ is an isometric unital $*$-homomorphism. Since G is an isometric unital $*$-isomorphism from B to $C(\operatorname{Sp} B)$, we deduce that $\Phi = G^{-1} \circ \Psi$ is an isometric unital $*$-homomorphism from $C(S)$ to B. In particular its image is closed. As we know that $\Phi(C(S))$ is contains the dense subalgebra $\mathcal{B}$ of B it follows that $\Phi(C(S)) = B$, i.e., Φ maps onto C^*-algebra generated by x.

As for the last part, combining Proposition I.6 with the fact that Φ is an isomorphism of algebras from $C(S)$ onto B shows that, for all $f \in C(S)$,

$$\operatorname{Sp}_A f(x) = \operatorname{Sp}_B f(x) = \operatorname{Sp}_{C(S)} f = f(S).$$

The proof is complete. □

REMARK I.17. The continuity of the map $f \to f(x)$ from $C(S)$ to A implies that if a sequence $(f_k) \subset \mathcal{P}$ converges uniformly on S to f, then

$$f(x) = \lim_{k \to \infty} f_k(x). \tag{I.5.4}$$

Together with (I.5.2) this provides us with an alternative, and somewhat more concrete, definition of $f(x)$.

REMARK I.18. Notice that the selfadjoint elements of $C(S)$ are exactly the continuous functions on S that are real-valued. Therefore, the fact that the continuous functional calculus is a $*$-homomorphism implies that, for any real-valued continuous function $f(z)$ on S, the element $f(x)$ is selfadjoint.

I.6. Examples of Noncommutative Quotients

I.6.1. Crossed-product algebra. Let A be a C^*-algebra and let G be a locally compact group together with a continuous action $(g, x) \to \alpha_g(x)$ of G on A.

A covariant representation of (A, G, α) in a Hilbert space $\mathcal{H}$ is a pair (π_A, π_G) such that:

- $\pi_A : A \to \mathcal{L}(\mathcal{H})$ is an isometric $*$-representation of A in $\mathcal{H}$;
- $\pi_G : G \to \mathcal{L}(\mathcal{H})$ is a unitary representation of G in $\mathcal{H}$, i.e., $\pi(g)$ is unitary for all $g \in G$;
- For all $x \in A$ and all $g \in G$,

$$\pi_G(g)\pi_A(x)\pi_G(g)^{-1} = \pi_A(\alpha_g(x)).$$

The covariant representation is said to be isometric when π_A is isometric. There always exists an isometric covariant representation of (A, G, α). For instance, given an isometric $*$-representation $\pi : A \to \mathcal{L}(\mathcal{H})$ of A into a Hilbert $\mathcal{H}$ as given by Theorem I.1, then we have an isometric covariant representation (π_A, π_G)

of (A, G, α) into the Hilbert space $L^2(G, \mathcal{H})$, with $\pi_A : A \to \mathcal{L}(L^2(G, \mathcal{H}))$ and $\pi_G : G \to \mathcal{L}(L^2(G, \mathcal{H}))$ given by

$$[\pi_A(x)\xi](h) := [\pi(\alpha_{h^{-1}}x)\xi](h) \qquad \forall x \in A \; \forall \xi \in L^2(G, \mathcal{H}) \; \forall h \in G,$$
$$[\pi_G(g)\xi](h) := \xi(g^{-1}h) \qquad \forall g, h \in G \; \forall \xi \in L^2(G, \mathcal{H}).$$

Consider the algebra $C_c(G, A)$ of continuous functions $f : G \to A$ with compact supports. We endow it with the convolution product,

$$f_1 * f_2(g) := \int_G f_1(h)\alpha_h \left[f_2(h^{-1}g)\right] d\lambda(g) \qquad \forall f_j \in C_c(G, A) \; \forall g \in G,$$

where $d\lambda$ is a left-invariant Haar measure on G (such a measure is unique up to constant multiple). We also equip $C_c(G, A)$ with the involution $f \to f^*$ given by

$$f^*(g) := \Delta(g)^{-1} f(g^{-1})^* \qquad \forall f \in C_c(G, A) \; \forall g \in G$$

where $\Delta(g)$ is the modular function of G. The latter is uniquely determined by the relation $d\lambda(gh^{-1}) = \Delta(h)d\lambda(g)$ $\forall h \in G$ or, equivalently, $d\lambda(g^{-1}) = \Delta(g)^{-1}d\lambda(g)$.

Remark I.19. If $\mu(x)$ is a Borel measure on G and $\phi : G \to G$ is a Borel function, then we denote by $\mu_\phi(x) = \mu(\phi(x))$ the Borel measure on G defined by

$$\mu_\phi(B) = \mu(\phi(B)) \qquad \text{for any Borel set } B.$$

For instance, if $G = \mathbb{R}$, the measure $\mu(x)$ is the Lebesgue measure dx and the function $\phi(x)$ is differentiable, then we have $d(\phi(x)) = \phi'(x)dx$.

Let (π_A, π_G) be a covariant representation of (A, G, α). We define a $*$-representation $\pi : C_c(G, A) \to \mathcal{L}(\mathcal{H})$ by letting

$$\pi(f) = \int_G \pi_A(f(g))\pi_G(g)d\lambda(g) \qquad \forall f \in C_c(G, A). \tag{I.6.1}$$

Then $\pi(C_c(G, A))$ is an involutive subalgebra of $\mathcal{L}(\mathcal{H})$.

Definition I.20. *The C^*-algebra obtained as the closure of $\pi(C_c(G, A))$ in $\mathcal{L}(\mathcal{H})$ is called the (reduced) crossed-product algebra of A by G and is denoted $A \rtimes_r G$.*

Up to $*$-isomorphism the reduced crossed-product algebra does not depend on the choice of the covariant representation (π_A, π_G) to define π.

I.6.2. Dual of a locally compact group. An important special case of the previous construction is the reduced C^*-algebra of the group G obtained when $A = \mathbb{C}$ and the action of G is trivial, i.e., $\alpha_g(x) = x$. This C^*-algebra is denoted $C^*_r(G)$. It can be realized by using the left-regular representation of G in $L^2(G)$, i.e., the unitary representation $\pi_G : G \to L^2(G)$ defined by

$$[\pi_G(g)\xi](h) = \xi(g^{-1}h) \qquad \forall g, h \in G \; \forall \xi \in L^2(G).$$

The representation (I.6.1) then is the representation of $C_c(G)$ in $L^2(G)$ such that, for all $f \in C_c(G)$ and all $\xi \in L^2(G)$ and $h \in G$, we have

$$[\pi(f)\xi](h) = \int_G f(g)\left[\pi_G(g)\xi\right](h)d\lambda(g) = \int_G f(g)\xi(g^{-1}h)d\lambda(g) = f * \xi(h),$$

where $*$ is the convolution for functions on G. Thus π is the representation of $C_c(G)$ in $L^2(G)$ by left-convolution operators. Its closure in $\mathcal{L}(L^2(G))$ is the C^*-algebra $C^*_r(G)$.

Assume now that G is Abelian, and let $\hat{G}$ be its Pontryagin dual. As $\hat{G}$ is Abelian $\hat{G}$ is set of characters of Γ, i.e., $\hat{G}$ consists of all continuous morphisms $\chi : G \to S^1$. Equipped with the pointwise product and the topology of convergence on compact sets, $\hat{G}$ is a locally compact group. The Fourier transform on G is the linear map $f \to \hat{f}$ from $L^1(G)$ to $C(\hat{G})$ defined by

$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}d\lambda(g) \qquad f \in L^1(G)\ \forall \chi \in \hat{G}.$$

For instance when $G = \mathbb{R}$ the characters are exactly those of the forms $\chi(t) = e^{ixt}$ for some $x \in \mathbb{R}$, so we recover the usual Fourier transform on $\mathbb{R}$.

The range of the Fourier transform is contained in $C_0(\hat{G})$. Moreover it extends to an isometric isomorphism from $L^2(G)$ onto $L^2(\hat{G})$ under which the convolution of functions on G corresponds to the pointwise product of functions on $\hat{G}$. In particular, for $f \in C_c(G)$ and $\xi \in L^2(G)$, we have

$$[\pi(f)\xi]^\wedge = (f * \xi)^\wedge = \hat{f}.\hat{\xi}.$$

Thus under the Fourier transform the representation π of $C_c(G)$ coincides with the representation of $C_0(\hat{G})$ by multiplication operators on $L^2(\hat{G})$. Since the latter is isometric we get an isometric $*$-homomorphism $\hat{\phi} : \pi(C_c(G)) \to C_0(\hat{G})$ such that

$$\hat{\phi}\,(\pi(f)) = \hat{f} \qquad \forall f \in C_c(G).$$

By definition $C_r^*(G)$ is the closure of $\pi(C_c(G))$ in $\mathcal{L}(L^2(G))$, so $\hat{\phi}$ immediately extends to an isometric $*$-homomorphism $\hat{\phi} : C_r^*(G) \to C_0(\hat{G})$. This homomorphism is onto. Indeed, for $\chi \in \hat{G}$ denote by ρ_χ the multiplication by χ^{-1} on $\hat{G}$. This gives rise to an action $(\chi, f) \to f \circ \rho_\chi$ of $\hat{G}$ on $C_c(\hat{G})$. This action preserves $\hat{\phi}(C_c(G))$, for we have

$$\hat{f}(\chi_1^{-1}\chi_2) = (\chi_1 f)^\wedge\,(\chi_2) \qquad \forall f \in C_c(G)\ \forall \chi_j \in \hat{G}.$$

This implies that $\hat{\phi}(C_c(G))$ separates the points of $\hat{G}$. Thus, we may use the Stone-Weierstrass theorem as in its version of [**Fo**, 4.52] to deduce that $\hat{\phi}(C_c(G))$ is dense in $C_0(G)$, and hence $\hat{\phi}(C_r^*(G)) = C_0(\hat{G})$. Therefore, we obtain:

Proposition I.12. *The Fourier transform on G gives rise to a $*$-isomorphism from $C_r^*(G)$ onto $C_0(G)$.*

When G is not Abelian the Pontryagin dual $\hat{G}$ is defined in terms of irreducible unitary representations. Its topology may not even be Hausdorff, so in general we cannot make use of techniques of classical topology to obtain much information on $\hat{G}$. However, the C^*-algebra $C_r^*(G)$ always makes sense and its representations are intimately related to the unitary representations of G. This motivates the study of the C^*-algebra $C_r^*(G)$ to gain information on G and its unitary representations.

I.6.3. Action of a Lie group on a manifold. Let G be a Lie group acting smoothly on a smooth manifold M. We write the action of G on M as a left-action $(g, x) \to g.x$. Then G acts continuously on the C^*-algebra $C_0(M)$ by

$$(g, f) \to \alpha_g(f), \qquad \alpha_g(f)(x) := f(g^{-1}.x) \quad \forall x \in M.$$

We equip M with a smooth measure $\rho(x)$ (in the terminology of (the forthcoming) Chapter 4 this is a density). We form the Hilbert space $L^2(M)$ using $d\rho(x)$.

We then have a natural isometric $*$-representation $\pi_1 : C_0(M) \to \mathcal{L}(L^2(M)$ given by the action of $C_0(M)$ on $L^2(M)$ by multiplication operators, i.e.,

$$\pi_1(f)\xi = f\xi \qquad \forall f \in C_0(M)\ \forall \xi \in L^2(M).$$

In addition, a unitary representation $\pi_2 : G \to L^2(M)$ is given by

$$\left[\pi_2(g)\xi\right](x) := \kappa_g(x)^{\frac{1}{2}}\xi(g^{-1}.x) \qquad \forall g \in G\ \forall \xi \in L^2(M)\ \forall x \in M,$$

where, for $g \in G$, we denote by $\kappa_g(x)$ the positive smooth function on M such that $d\rho(g.x) = \kappa_g(x)d\rho(x)$.

The pair (π_1, π_2) forms a covariant representation of $(C_0(M), G, \alpha)$, using which we can define an isometric $*$-representation $\pi : C_c(G, C_0(M)) \to \mathcal{L}(L^2(M))$ as in (I.6.1). It is useful to consider the dense subalgebra $C_c(G, C_c(M))$ and to identify it with $C_c(G \times M)$. Then for $f \in C_c(G \times M)$ the operator $\pi(f) \in \mathcal{L}(L^2(M))$ is simply given by

$$\left[\pi(f)\xi\right](x) = \int_G f(g,x)\xi(g^{-1}.x)\kappa_g(x)^{\frac{1}{2}}d\lambda(g) \qquad \forall \xi \in L^2(M)\ \forall x \in M.$$

Then the crossed-product algebra $C_0(M) \rtimes_r G$ is the C^*-algebra obtained as the closure in $\mathcal{L}(L^2(M))$ of $\pi(C_c(G\times M))$, where $\pi : C_c(G\times M) \to \mathcal{L}(L^2(M))$ is defined as above.

The action of G on M is said to be *free* if every $g \in G \setminus 1$ has no fixed points, and it is said to be *proper* if the map $(g, x) \to (x, g.x)$ is a proper map from $G \times M$ to $M \times M$. If the action is free and proper then the quotient topology of the orbit space M/G is Hausdorff and M/G can be endowed with the unique smooth manifold structure such that the canonical surjection $M \to M/G$ is a smooth map (see, e.g., [**Ca**, Thm. 23.4]).

Assuming the action of G to be free and proper, we denote by $\mathcal{H}$ the bundle of Hilbert spaces over M/G whose fiber at $w = Gx$ is the L^2-space $L^2(Gx)$ formed by pushing-forward to Gx the left-Haar measure $d\lambda(g)$ under the map $g \to g.x$. Denoting by $\mathcal{K}(\mathcal{H})$ the bundle of C^*-algebras over M/G whose fiber at $w \in M/G$ is the C^*-algebra $\mathcal{K}(\mathcal{H}_w)$ of compact operators on $\mathcal{H}_w$, the space $C_0(M/G, \mathcal{K}(\mathcal{H}))$ of continuous sections of $\mathcal{K}(\mathcal{H})$ vanishing at ∞ is a C^*-algebra; this is the C^*-algebra generated by the bundle $\mathcal{H}$.

Proposition I.13 (Green [**Gr**]). *If the action of G on M is free and proper, then we have an (isometric) $*$-isomorphism,*

$$C_0(M) \rtimes_r G \simeq C_0(M/G, \mathcal{K}(\mathcal{H})). \tag{I.6.2}$$

While $C_0(M/G, \mathcal{K}(\mathcal{H}))$ may be complicated to describe, a consequence of Green's result is the strong Morita equivalence,

$$C_0(M) \rtimes_r G \underset{\text{M.E.}}{\simeq} C_0(M/G).$$

Two algebras A and B are said to be Morita equivalent if there is an $(A-B)$-bimodule $\mathcal{M}_1$ and a $(B-A)$-bimodule $\mathcal{M}_2$ such that as an A-module isomorphism $\mathcal{M}_1 \otimes_B \mathcal{M}_2 \simeq A$ and a B-module isomorphism $\mathcal{M}_2 \otimes_A \mathcal{M}_1 \simeq B$. The notion of strong Morita equivalence between C^*-algebras, due to Rieffel, is the equivalent notion in terms of C^*-bimodules over C^*-algebras. We refer to [**Co**, Appendix A, Chap. 2] and to [**GVF**, Sect. 4.5] for the main definitions and properties regarding strong Morita equivalence of C^*-algebras. For instance, the strong Morita equivalence between $C_0(M/G)$ and $C_0(M/G, \mathcal{K}(\mathcal{H}))$ is realized by $C_0(M/G, \mathcal{H})$ seen as a $C_0(M/G) - C_0(M/G, \mathcal{K}(\mathcal{H}))$ C^*-bimodule.

Although strong Morita equivalence is a weaker notion of equivalence than that provided by $*$-isomorphisms, many properties of the latter continue to hold for Morita equivalence. For instance two Morita-equivalent C^*-algebras have same representation theory, and anticipating the second half of the course, they have same K-theory and cyclic cohomology.

When the action of G is not free the quotient space M/G need not even be Hausdorff, and so we cannot make use of classical differential geometry to study it. However, the crossed-product always makes well sense, so you can study it to get geometric information on the action of G on M. This simple example is of one of the main motivation for extending the tools of differential geometry into the operator theoretic framework of noncommutative geometry.

I.6.4. Noncommutative torus. Consider the 1-dimensional torus,

$$\mathbb{T} = S^1 = \{z \in \mathbb{C};\ |z| = 1\}.$$

We represent the C^*-algebra $A = C(\mathbb{T})$ by multiplication operators on $L^2(\mathbb{T})$, that is, by using the representation $\pi_{C(\mathbb{T})} : C(\mathbb{T}) \to \mathcal{L}(L^2(\mathbb{T}))$ given by

$$\pi_1(f)g = fg \qquad \forall f \in C(\mathbb{T})\ \forall g \in L^2(\mathbb{T}).$$

Let $\theta \in \mathbb{R}$. This defines an action of $\mathbb{Z}$ on $\mathbb{T}$ by

$$k.z := e^{-2ik\pi\theta} z \qquad \forall (k, z) \in \mathbb{Z} \times \mathbb{T}. \tag{I.6.3}$$

This yields an action of $\mathbb{Z}$ on $C(\mathbb{T})$ by

$$(k, f) \to \alpha_k(f), \qquad [\alpha_k(f)](z) = f(e^{2ik\pi\theta} z). \tag{I.6.4}$$

This also gives rise the unitary representation $\pi_{\mathbb{Z}} : \mathbb{Z} \to \mathcal{L}(L^2(\mathbb{T}))$ defined by

$$\pi_2(f)(z) = f(e^{2ik\pi\theta} z) \qquad \forall f \in L^2(\mathbb{T})\ \forall z \in \mathbb{T}.$$

Then the pair (π_1, π_2) is an isometric covariant representation of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ in $L^2(\mathbb{T})$.

Definition I.21. *The noncommutative torus, denoted A_θ, is the crossed-product C^*-algebra $C(\mathbb{T}) \rtimes_r \mathbb{Z}$, where the action of $\mathbb{Z}$ on $C(\mathbb{T})$ is given by (I.6.4).*

Let us now work out explicitly the construction of $C(\mathbb{T}) \rtimes_r \mathbb{Z}$. Since $\mathbb{Z}$ is discrete, $C_c(\mathbb{Z}, C(\mathbb{T}))$ consists of finite sequences $(f_k)_{k\in\mathbb{Z}}$ with values in $C(\mathbb{T})$. Let $f = (f_k)_{k\in\mathbb{Z}}$ be such a sequence and let us further assume that each function f_k is a Fourier polynomial on $\mathbb{T}$, i.e., $f_k = \sum_{l\in\mathbb{Z}} a_{kl} z^l$, $a_{kl} \in \mathbb{C}$, where the sum is finite.

Since $\mathbb{Z}$ is discrete, as left-Haar measure we can take the counting measure. Then Eq. (I.6.1) becomes

$$\pi(f) = \sum_k \pi_1(f_k)\pi_2(k) = \sum_k \pi_1\left(\sum_l a_{kl} z^l\right)\pi_2(k.1) = \sum_{k,l} a_{kl}\pi_1(z)^l \pi_2(1)^k. \tag{I.6.5}$$

Set $U = \pi_1(z)$ and $V = \pi_2(1)$. These are the unitaries of $L^2(\mathbb{T})$ given by

$$Uf(z) = zf(z) \qquad \text{and} \qquad Vf(z) = f(e^{2i\pi\theta} z).$$

Moreover, we have

$$VU = e^{2i\pi\theta} UV.$$

From (I.6.5) we see that $\pi(f) = \sum_{k,l} a_{kl} U^k V^l$. Using the density of polynomials in z and $\bar{z} = z^{-1}$ in $C(\mathbb{T})$ we obtain:

Proposition I.14. *A_θ is the sub-C^*-algebra of $\mathcal{L}(L^2(\mathbb{T}))$ generated by the unitaries U and V with relation $VU = e^{2i\pi\theta} UV$.*

It can be shown that any C^*-algebra generated by unitaries u and v with relation $vu = e^{2i\pi\theta} uv$ is $*$-isomorphic to A_θ (see [**GVF**, Prop. 12.1]). In particular, if $\theta \in \mathbb{Z}$ then there is an isometric $*$-isomorphism,

$$A_\theta \simeq C(\mathbb{T}^2).$$

In fact, the following holds.

Proposition I.15 (Rieffel; see [**GVF**, Prop. 12.2]). *If $\theta = \frac{p}{q}$, where p and q are relatively prime integers, then there is an isometric $*$-isomorphism,*

$$A_\theta \simeq C(\mathbb{T}^2, \mathcal{E}).$$

where $\mathcal{E}$ is a bundle of algebras over $\mathbb{T}^2$ with typical fiber $M_q(\mathbb{C})$. In particular A_θ is Morita equivalent to $C(\mathbb{T}^2)$.

For $\theta \notin \mathbb{Q}$ the noncommutative torus is one of the most basic examples of a truly noncommutative space. We refer to [**Co**, Sect. 8, Chap. 2] and to [**GVF**, Sect. 12] for more detailed accounts on it.

Bibliography

[Ar] Arveson, W.: *A Short course on spectral theory.* Graduate Texts in Mathematics, 209. Springer-Verlag, Berlin-New York, 2002.

[Ca] Cannas da Silva, A.: *Lectures on symplectic geometry.* Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001

[Co] Connes, A.: *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[Gr] Green, P.: *C^*-algebras of transformation groups with smooth orbit spaces.* Pacific J. Math. **72** (1977) 71–97.

[GVF] Gracia-Bondía, J.M.; Várilly, J.C.; Figueroa, H.: *Elements of noncommutative geometry.* Birkhäuser Boston, Boston, MA, 2001.

[Fo] Folland, G.: *Real analysis.* 2nd edition. John Wiley & Sons, Inc., New York, 1999.

[Yo] Yosida, K.: *Functional analysis.* 6th edition. Grundlehren der Mathematischen Wissenschaften, 123. Springer-Verlag, Berlin-New York, 1980.

CHAPTER II

Operators on a Hilbert Space

This chapter is a review of basic results concerning operators on a Hilbert space. The main reference for this chapter is the book of Reed-Simon [**RS**].

Throughout this chapter we let $\mathcal{H}$ be a separable Hilbert space and we denote by $\mathcal{L}(\mathcal{H})$ its C^*-algebra of continuous endomorphisms.

II.1. Polar Decomposition

DEFINITION II.1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be positive if can be written in the form $T = S^*S$ with $S \in \mathcal{L}(\mathcal{H})$.*

We denote by $\mathcal{L}(\mathcal{H})_+$ the set of positive elements of $\mathcal{L}(\mathcal{H})$.

PROPOSITION II.1. *For $T \in \mathcal{L}(\mathcal{H})$ the following are equivalent:*

(i) *T is positive.*
(ii) *There exists $S \in \mathcal{L}(\mathcal{H})$ selfadjoint such that $T = S^2$.*
(iii) *T is selfadjoint and $\operatorname{Sp} T \subset [0, \infty)$.*
(iv) *$\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$.*

PROOF. We shall prove the implications (i) $\Rightarrow$ (iv), (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii), which will give the proposition.

• (i) $\Rightarrow$ (iv): If $T = S^*S$ for some $S \in \mathcal{L}(\mathcal{H})$ then, for all $\xi \in \mathcal{H}$,

$$\langle T\xi, \xi\rangle = \langle S^*S\xi, \xi\rangle = \langle S\xi, S\xi\rangle = \|S\xi\|^2 \geq 0,$$

proving that (i) $\Rightarrow$ (iv).

• (ii) $\Rightarrow$ (i): If $T = S^2$ with $S \in \mathcal{L}(\mathcal{H})$ selfadjoint, then $T = S^*S$ and hence T is positive. Thus (ii) $\Rightarrow$ (i).

• (iii) $\Rightarrow$ (ii): Assume that T is selfadjoint and $\operatorname{Sp} T \subset [0, \infty)$. Then the function $\sqrt{t}$ is defined and continuous on $\operatorname{Sp} T$, and so we can define $S := \sqrt{T}$ by functional continuous calculus. Notice that, as $\sqrt{t}$ takes real values on $\operatorname{Sp} T$, it follows from Remark I.18 that S is selfdajoint. Recall that the continuous functional calculus $f \to f(T)$ is a homomorphism of algebrs from $C(\operatorname{Sp} T)$ to $\mathcal{L}(\mathcal{H})$. Since $(\sqrt{t})^2 = t$ on $\operatorname{Sp} T$, it follows that $S^2 = (\sqrt{T})^2 = T$, and hence T satisfies (ii). Thus (iii) $\Rightarrow$ (ii).

• (iv) $\Rightarrow$ (iii): Assume that $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. Then T is selfadjoint. Indeed, for all ξ, η in $\mathcal{H}$, we have

$$4\langle T\xi, \eta\rangle = \langle T(\xi+\eta), \xi+\eta\rangle - \langle T(\xi-\eta), \xi-\eta\rangle - i\langle T(\xi+i\eta), \xi+i\eta\rangle + i\langle T(\xi-i\eta), \xi-i\eta\rangle,$$

from which we see that $\overline{\langle T\xi, \eta\rangle} = \langle T\eta, \xi\rangle$, and hence $\langle T\xi, \eta\rangle = \langle \xi, T\eta\rangle$.

Let us now show that $\operatorname{Sp} T \subset [0, \infty)$. Since T is selfadjoint, and hence $\operatorname{Sp} T \subset \mathbb{R}$ by Proposition I.7, we only need to show that $T - \lambda$ is invertible for all $\lambda < 0$. Thus let $\lambda \in (-\infty, 0)$ and let $\xi \in \mathcal{H}$. As $\langle T\xi, \xi\rangle \geq 0$ we have

$$|\lambda|\|\xi\|^2 = -\lambda\langle \xi, \xi\rangle \leq \langle (T-\lambda)\xi, \xi\rangle \leq \|(T-\lambda)\xi\|\|\xi\|.$$

Thus,

$$\|(T-\lambda)\xi\| \leq |\lambda|\|\xi\| \qquad \forall \xi \in \mathcal{H}. \tag{II.1.1}$$

This implies that $\ker(T-\lambda) = \{0\}$. As $T-\lambda$ is selfadjoint we then see that $\overline{\operatorname{im}(T-\lambda)} = (\ker(T-\lambda))^{\perp} = \mathcal{H}$.

In fact, the inequality (II.1.1) also implies that $\operatorname{im}(T-\lambda)$ is closed. Indeed, if $\eta = \lim(T-\lambda)\xi_n$, then (II.1.1) implies that the sequence $(\xi_n)_{n\geq 0}$ is Cauchy in $\mathcal{H}$, and hence ξ_n converges to some ξ in $\mathcal{H}$. Then $\eta = \lim(T-\lambda)\xi_n = T\xi$, showing that η is contained in $\operatorname{im}(T-\lambda)$. Thus $\ker(T-\lambda) = 0$ and $\operatorname{im}(T-\lambda) = \mathcal{H}$, i.e., $T-\lambda$ is bijective. Recall that by the open mapping theorem any bijective continuous linear map of $\mathcal{H}$ onto $\mathcal{H}$ has a continuous inverse (see [**Fo**, p. 162]), so $T-\lambda$ is an invertible element of $\mathcal{L}(\mathcal{H})$. Thus $\operatorname{Sp} T \subset [0, \infty)$, and hence T satisfies (iii), showing that (iv) implies (iii). The proof is complete. □

Corollary II.1. *The set of positive operators $\mathcal{L}(\mathcal{H})_+$ is a positive cone of $\mathcal{L}(\mathcal{H})$, i.e.,*

$$\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{L}(\mathcal{H})_+ \qquad \forall T_j \in \mathcal{L}(\mathcal{H})_+ \ \forall \lambda_j \geq 0.$$

Proof. For $j = 1, 2$ let $T_j \in \mathcal{L}(\mathcal{H})_+$ and let $\lambda_j \in [0, \infty)$. Using the characterization (iv) of Proposition II.1, we see that, for all $\xi \in \mathcal{H}$,

$$\langle (\lambda_1 T_1 + \lambda_2 T_2)\xi, \xi\rangle = \lambda_1\langle T_1\xi, \xi\rangle + \lambda_2\langle T_2\xi, \xi\rangle \geq 0,$$

proving that $\lambda_1 T_1 + \lambda_2 T_2$ is a positive operator. □

Corollary II.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be normal and let $f \in C(\operatorname{Sp} T)$ be non-negative. Then the operator $f(T)$ is positive.*

Proof. Since f is real-valued it follows from Remark I.18 that $f(T)$ is selfadjoint. Moreover, by (I.5.3) we have $\operatorname{Sp} f(T) = f(\operatorname{Sp} T) \subset [0, \infty)$, so it follows from Proposition II.1 (iii) that $f(T)$ is positive. □

Let $T \in \mathcal{L}(\mathcal{H})$. Then T^*T is a positive operator, so by the previous proposition T^*T is selfadjoint and its spectrum is contained in $[0, \infty)$. Therefore, by continuous functional calculus we can define its square root $\sqrt{T^*T}$ as an element of $\mathcal{L}(\mathcal{H})$. It follows from Corollary II.2 that $|T|$ is a positive operator.

Definition II.2. *For all $T \in \mathcal{L}(\mathcal{H})$ the operator $\sqrt{T^*T}$ is denoted $|T|$ and is called the modulus of T.*

Lemma II.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

(i) *$|T|$ is the unique element of $\mathcal{L}(\mathcal{H})_+$ whose square is T^*T.*
(ii) *We have*

$$\||T|\xi\| = \|T\xi\| \qquad \forall \xi \in \mathcal{H}, \tag{II.1.2}$$

and hence $\ker |T| = \ker T$.

Proof. The continuous functional calculus $f \to f(T^*T)$ is a $*$-homomorphism from $C(\operatorname{Sp} T^*T)$ to $\mathcal{L}(\mathcal{H})$. As $(\sqrt{t})^2 = t$ on $[0, \infty)$ it follows that $|T|^2 = (\sqrt{T^*T})^2 = T^*T$.

Let $S \in \mathcal{L}(\mathcal{H})_+$ be such that $S^2 = T$. Since S is positive, by Proposition II.1 its spectrum is contained in $[0, \infty)$, and hence $\sqrt{t^2} = t$ on $\operatorname{Sp} S$. Therefore, by continuous functional calculus $S = \sqrt{S^2} = \sqrt{T^*T} = |T|$. Thus $|T|$ is the unique element of $\mathcal{L}(\mathcal{H})_+$ whose square is T^*T.

As $|T|$ is selfadjoint $|T|^*|T| = |T|^2 = T^*T$. Therefore, for all $\xi \in \mathcal{H}$,

$$\||T|\xi\|^2 = \langle |T|\xi, |T||\xi|\rangle = \langle |T|^*|T|\xi, \xi\rangle = \langle T^*T\xi, \xi\rangle = \langle T\xi, T\xi\rangle = \|T\xi\|^2,$$

proving (II.1.2). This immediately implies that $|T|$ and T have same kernel. ☐

Proposition II.2 (Polar Decomposition). *Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a unique $U \in \mathcal{L}(\mathcal{H})$, called the phase of T, such that*

(i) *$T = U|T|$;*

(ii) *$\ker U = \ker |T|$.*

Proof. Since $|T$ is selfadjoint $(\ker |T|)^\perp = \overline{\operatorname{im} |T|}$, and hence $|T|$ is a bijection from $\operatorname{im} |T|$ onto itself. Let $|T|^{-1} : \operatorname{im} |T| \rightarrow \operatorname{im} |T|$ be its inverse and denote by U the linear map $T|T|^{-1} : \operatorname{im} |T| \rightarrow \mathcal{H}$. Then, for any $\xi \in \operatorname{im} |T|$, we have

$$\begin{aligned}\|U\xi\|^2 = \langle T|T|^{-1}\xi, T|T|^{-1}\xi\rangle = \langle T^*T|T|^{-1}\xi, |T|^{-1}\xi\rangle = \langle |T|^2|T|^{-1}\xi, |T|^{-1}\xi\rangle \\ = \langle |T|\xi, |T|^{-1}\xi\rangle = \langle \xi, |T||T|^{-1}\xi\rangle = \|\xi\|^2.\end{aligned}$$

Thus U uniquely extends to an isometric linear map $U : \overline{\operatorname{im} |T|} \rightarrow \mathcal{H}$. Extending U to be 0 on $\ker |T| = (\operatorname{im} |T|)^\perp$ we then get a continuous endomorphism $U : \mathcal{H} \rightarrow \mathcal{H}$ whose null space is $\ker |T|$.

If $\xi \in \operatorname{im} |T|$, then $U|T|\xi = T|T|^{-1}|T|\xi = \xi$, and hence $T = U|T|$ on $\overline{\operatorname{im} |T|}$ by continuity. Moreover, as by Lemma II.1 $\ker |T| = \ker T$, we have $U|T|\xi = 0 = T\xi$ for all ξ in $\ker |T| = (\operatorname{im} |T|)^\perp$. Therefore, we see that $T = U|T|$ on $\mathcal{H}$, showing that U satisfies the conditions (i) and (ii) of the proposition.

Let $V \in \mathcal{L}(\mathcal{H})$ be such that $T = V|T|$ and $\ker V = \ker |T|$. If $\xi \in \ker |T|$, then obviously $V\xi = 0 = U\xi$. If $\xi \in \operatorname{im} |T|$, then $V\xi = V|T||T|^{-1}\xi = T|T|^{-1}\xi = U\xi$, so by continuity $V = U$ on $\overline{\operatorname{im} |T|}$. It follows from this that $V = U$ on $\mathcal{H}$, and hence U is the unique element of $\mathcal{L}(\mathcal{H})$ such that $T = U|T|$ and $\ker U = \ker |T|$, giving the proposition. ☐

Proposition II.3. *Let $T \in \mathcal{L}(\mathcal{H})$ have polar decomposition $T = U|T|$ and denote by $\Pi_0(T)$ (resp. $\Pi_0(T^*)$) the orthogonal projection onto $\ker T$ (resp. $\ker T^*$).*

(i) *The range of U is $\overline{\operatorname{im} T}$.*

(ii) *We have*

$$U^*U = 1 - \Pi_0(T) \qquad \text{and} \qquad UU^* = 1 - \Pi_0(T^*),$$

so that U is a partial isometry and has norm 1 unless $T = 0$.

(iii) *If T is injective and has dense range, then U is unitary.*

(iv) *We have*

$$|T| = U^*T, \qquad T^* = UTU^*, \qquad |T^*| = TU^* = U|T|U^*.$$

(v) *The phase of T^* is U^*.*

Proof. As U vanishes on $\ker |T|$ we see that $\operatorname{im} U = U(\ker |T|)$. Notice that by its construction in the proof of Proposition II.2 the operator U is isometric on $(\ker |T|)^\perp$ and agrees with $T|T|^{-1}$ on $\operatorname{im} |T|$. In particular, it follows from Lemma I.1 that $U((\ker |T|)^\perp)$ is closed and U induces a unitary operator from

$(\ker|T|)^\perp$ onto $U((\ker|T|)^\perp) = \operatorname{im} U$. As $(\ker|T|)^\perp = \overline{\operatorname{im} T}$ we then see that $\operatorname{im} U = \overline{U(\operatorname{im}|T|)}$. Since $U = T|T|^{-1}$ on $\operatorname{im}|T|$ we have $U(\operatorname{im}|T|) \subset \operatorname{im} T$, and hence $\operatorname{im} U \subset \overline{\operatorname{im} T}$. However, as $T = U|T|$ we also have $\operatorname{im} T \subset \operatorname{im} U$, and as $\operatorname{im} U$ is closed we see that $\overline{\operatorname{im} T}$ is contained is $\operatorname{im} U$, and so the range of U is $\overline{\operatorname{im} T}$.

As abovementioned U induces a unitary operator from $(\ker|T|)^\perp$ onto $\operatorname{im} U = \overline{\operatorname{im} T}$. Since by Lemma II.1 $\ker|T| = \ker T$ this shows that U is a unitary endomorphism of $\mathcal{H}$ when T is injective and has dense range. In any case, as we have $\overline{\operatorname{im} T} = (\ker T^*)^\perp$ we see that U induces a unitary operator from $(\ker T)^\perp$ onto $(\ker T^*)^\perp$. Since $\ker U = \ker|T| = \ker T$ we then deduce that U^*U is the orthogonal projection onto $(\ker T)^\perp$ and UU^* is the orthogonal projection onto $(\ker T)^\perp$, that is, $U^*U = 1 - \Pi_0(T)$ and $UU^* = 1 - \Pi_0(T^*)$. In particular, if $T \neq 0$ then $\|U\|^2 = \|U^*U\| = \|1 - \Pi_0(T)\| = 1$, i.e., $\|U\| = 1$.

Notice that, as $\ker|T| = \ker T$, we have $|T|(1-\Pi_0(T)) = (1-\Pi_0(T))|T| = |T|$, and hence $U^*T = U^*U|T| = (1-\Pi_0(T))|T| = |T|$. Moreover,

$$(U|T|U^*)^2 = U|T|U^*U|T|U^* = U|T|(1-\Pi_0(T))|T|U^* = (U|T|)(U|T|)^* = TT^*.$$

As $|T|$ is positive, for any $\xi \in \mathcal{H}$, we have $\langle U|T|U^*\xi, \xi\rangle = \langle |T|U^*\xi, U^*\xi\rangle \geq 0$, and hence $U|T|U^*$ is positive. Thus $U|T|U^*$ is a positive operator whose square is equal to TT^*, so using Lemma II.1 we see that $U|T|U^* = |T^*|$. Since $T = U|T|$ this also shows that $|T^*| = TU^*$.

Notice that $\ker U^* = (\operatorname{im} U)^\perp = (\operatorname{im} T)^\perp = \ker T^* = \ker|T^*|$. Moreover,

$$U^*|T^*| = U^*U|T|U^* = (1-\Pi_0(T))|T|U^* = |T|U^* = (U|T|)^* = T^*,$$

so by Proposition II.2 the phase of T^* is U^*. Notice that the above equalities include the equality $T^* = |T|U^*$. As $|T| = U^*T$ we see that $T^* = U^*TU^*$, completing the proof. ☐

II.2. Spectral Theorem and Borel Functional Calculus

Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator (i.e., $T^*T = TT^*$) and set $S = \operatorname{Sp} T$.

Theorem II.1 (Spectral Theorem; see [**RS**, Thm. VII.3]). *There exist a finite measure space (X,μ), a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$, and a bounded measurable function f on X, in such way that*

$$UTU^*\xi = f\xi \qquad \forall \xi \in L^2_\mu(X). \tag{II.2.1}$$

For $F \in \mathcal{L}^\infty(X)$ denote by T_F the multiplication operator by F, i.e., the operator $T_F \in \mathcal{L}(L^2_\mu(X))$ defined by

$$T_F\xi = F\xi \qquad \xi \in L^2_\mu(X).$$

For instance, Eq. (II.2.1) says that $UTU^* = T_f$.

The *essential range* of F consists of all $\lambda \in \mathbb{C}$ such that

$$\mu(\lambda - \epsilon < F < \lambda + \epsilon) > 0 \qquad \forall \epsilon > 0.$$

It can be shown that $\operatorname{Sp} T_F$ agrees with the essential range of F. In particular, we see that the essential range of T_f is S. Thus, without any loss of generality, we may assume that the range of f is S.

We endow $L^\infty(X)$ with its usual norm, i.e.,

$$\|F\|_{L^\infty(X)} = \{\lambda;\ \lambda \text{ in the essential range of } |F|\} \qquad \forall F \in L^\infty(\mathbb{R}).$$

We also endow $L^\infty(X)$ with the involution $F \to \overline{F}$ given by complex conjugation. This turns $L^\infty(X)$ into a commutative unital C^*-algebra. Then it is not difficult to check that the map $F \to T_F$ is a $*$-homomorphism from $L^\infty(X)$ to $\mathcal{L}(L^2_\mu(X))$. Moreover, as T_F is a normal operator, we have

$$\|T_F\| = \sup_{\lambda \in \operatorname{Sp} T_F} |\lambda| = \|F\|_{L^\infty(S)}.$$

If $g \in C(S)$ then $g \circ f$ is again a bounded measurable function on X. In fact, the continuous functional calculus for f is just $g \to g \circ f$. Since the maps $F \to T_F$ (resp., $F \to U^* T_F U$) is an isometric $*$-homomorphisms from $L^\infty(X)$ to $\mathcal{L}(L^2_\mu(X))$ (resp., $\mathcal{L}(\mathcal{H})$), it follows that, for all $g \in C(S)$,

$$g(T) = g(U^* T_f U) = U^* g(T_F) U = U^* T_{g\circ f} U. \tag{II.2.2}$$

We then can extend the definition of $g(T)$ for any any bounded Borel function g on $\operatorname{Sp} T$ by letting

$$g(T) := U^* T_{g \circ f} U.$$

This defines a bounded operator on $\mathcal{H}$.

Theorem II.2 (Borel Functional Calculus; see [**RS**, Thm. VII.2]). *The following hold.*

(1) *The map $g \to g(T)$ is a $*$-homomorphism from $L^\infty(S)$ to $\mathcal{L}(\mathcal{H})$ such that*

$$\|g(T)\| \leq \|g\|_{L^\infty(S)} \qquad \forall g \in L^\infty(S). \tag{II.2.3}$$

(2) *If $(g_n)_{n\geq 0}$ is a sequence of (bounded) Borel functions on S such that $g_n \to g$ a.e. and there exists $C > 0$ such that $\|g_n\|_{L^\infty(S)} \leq C$ for all $n \in \mathbb{N}_0$, then $g_n(T) \to g(T)$ strongly (i.e., $g_n(T)\xi \to g(T)\xi$ for all $\xi \in \mathcal{H}$).*

(3) *If $g \in L^\infty(S)$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).*

The spectral theorem and the Borel functional calculus can be extended to unbounded operators as follows.

Definition II.3. *An (unbounded) operator on $\mathcal{H}$ is a linear operator $T : D(T) \to \mathcal{H}$, where the domain $D(T)$ is a subspace of $\mathcal{H}$. T*

An operator T on $\mathcal{H}$ is said to be *densily defined* when its domain $D(T)$ is a dense subspace of $\mathcal{H}$. An operator S is said to be an extension of T, and we write $T \subset S$, if $D(T) \subset D(S)$ and S agrees with T on $D(T)$.

The *graph* of an operator T is defined to be

$$G(T) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \eta = T\xi\}.$$

The graph of T is a subspace of $\mathcal{H} \oplus \mathcal{H}$. We say that T is *closed* when $G(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.

If T is densily defined, then its *adjoint* is the operator T^* with graph

$$G(T^*) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \langle T\zeta, \xi\rangle = \langle \zeta, \eta\rangle\ \forall \zeta \in D(T)\}.$$

Since $G(T^*)$ is a closed subspace, we see that T^* is always a closed operator.

The *resolvent set* of T consists of all $\lambda \in \mathbb{C}$ such that

(i) $T - \lambda : D(T) \to \mathcal{H}$ is a bijection.
(ii) The inverse $(T - \lambda)^{-1} : \mathcal{H} \to D(T)$ is bounded.

The *spectrum* of T, denoted $\operatorname{Sp} T$, is the complement of the resolvent set.

It can be shown that $\operatorname{Sp} T$ is a closed subset of $\mathbb{C}$ (which may be empty) and the resolvent $\lambda \to (T-\lambda)^{-1}$ is analytic from $\mathbb{C} \setminus \operatorname{Sp} T$ to $\mathcal{L}(\mathcal{H})$ (provided we regard the inverses $(T-\lambda)^{-1}$ as elements of $\mathcal{L}(\mathcal{H})$).

A densily defined operator T is said to be *symmetric* is $T \subset T^*$. It is said to be *selfadjoint* if $T = T^*$.

PROPOSITION II.4 ([**RS**, Thm. VIII.3]). *Let T be a symmetric operator on $\mathcal{H}$. Then the following are equivalent:*

(i) *T is selfadjoint.*
(ii) *T is closed and* $\ker(T \pm i) = \{0\}$.
(iii) $\operatorname{im}(T \pm i) = \mathcal{H}$.

Let T be a selfadjoint (unbounded) operator on $\mathcal{H}$.

THEOREM II.3 (Spectral Theorem; see [**RS**, Thm. VIII.4]). *There exist a finite measure space (X, μ), a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$, and an a.e. bounded measurable real-valued function f on X, in such way that*

$$U\left(D(T)\right) = \{\xi \in L^2_\mu(X);\ f\xi \in L^2_\mu(X)\},$$
$$UTU^*\xi = f\xi \qquad \xi \in U\left(D(T)\right).$$

If g is a bounded Borel function on $\mathbb{R}$, then we define $g(T)$ as the bounded operator on $\mathcal{L}(\mathcal{H})$ given by

$$g(T) := U^* T_{g \circ f} U.$$

THEOREM II.4 (Borel Functional Calculus; [**RS**, Thm. VIII.5]). *The following hold.*

(1) *The map $g \to g(T)$ is a $*$-homomorphism from $L^\infty(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that*

$$\|g(T)\| \leq \|g\|_{L^\infty(\mathbb{R})} \qquad \forall g \in L^\infty(\mathbb{R}). \tag{II.2.4}$$

(2) *If $(g_n)_{n \geq 0}$ is a sequence of bounded Borel functions on $\mathbb{R}$ such that $g_n \to g$ a.e. and there exists $C > 0$ such that $\|g_n\|_{L^\infty(\mathbb{R})} \leq C$ for all $n \in \mathbb{N}_0$, then $g_n(T) \to g(T)$ strongly.*
(3) *If $g \in L^\infty(\mathbb{R})$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).*

More generally, if g is a possibly unbounded Borel function on $\mathbb{R}$, then $g(T)$ makes sense as an unbounded operator as follows. The domain of $g(T)$ is

$$D(g(T)) = U^*\bigg(\{\xi \in L^2_\mu(X);\ (g \circ f)\xi \in L^2_\mu(X)\}\bigg),$$

and we define $g(T)$ by the formula

$$g(T)\xi := U^*\left((g \circ f)U\xi\right) \qquad \forall \xi \in D(g(T)).$$

For instance, if $g(t) = t$, then $g(T) = T$. In addition, it is not hard to check that, if (g_n) be a sequence of Borel functions on $\mathbb{R}$ such that $g_n \to g$ and $|g_n| \leq |g|$, then as $n \to \infty$ we have

$$g_n(T)\xi \longrightarrow g(T)\xi \qquad \forall \xi \in D(g(T)).$$

EXAMPLE II.4. Let $\Delta = -(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2)$ be the (positive) Laplacian on $\mathbb{R}^n$. We shall regard Δ as an operator on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$. Denoting by $u \to \hat{u}$ the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, we have

$$(\Delta u)^\wedge = |\xi|^2 \hat{u} \qquad \forall u \in \mathcal{S}'(\mathbb{R}^n). \tag{II.2.5}$$

Thus, we can regard Δ as an unbounded operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ with domain,

$$D(\Delta) = \{u \in L^2(\mathbb{R}^n);\ |\xi|^2 \hat{u} \in L^2(\mathbb{R}^n)\}.$$

Notice that $D(\Delta)$ agrees with the Sobolev space $W^{2,2}(\mathbb{R}^n)$. As such Δ is a selfadjoint operator with spectrum $[0, \infty)$.

Denote by U the unitary operator on $L^2(\mathbb{R}^n)$ defined by

$$Uv = (2\pi)^{-\frac{n}{2}} \hat{v} \qquad \forall v \in L^2(\mathbb{R}^n).$$

Then it follows from (II.2.5) that

$$\Delta v = U^* T_{|\xi|^2} U v \qquad \forall v \in W^{2,2}(\mathbb{R}^n).$$

Therefore, we see that in this example the spectral theorem follows from elementary Fourier-analytic considerations.

If g is a bounded Borel function on $\mathbb{R}$, then $g(\Delta)$ is given by

$$g(\Delta)v(x) = (U^* T_{g(|\xi|^2)} U v)(x) = (2\pi)^{-n} \int e^{ix.\xi} g(|\xi|^2) \hat{u}(\xi) d\xi \quad \forall v \in L^2(\mathbb{R}^n).$$

The following operators are of special interest:

- The heat semigroup $e^{-t\Delta}$, $t \geq 0$.
- The wave group $e^{it\Delta}$, $t \in \mathbb{R}$.
- The complex powers Δ^z, $\Re z \geq 0$.

These operators can be similarly defined by Borel functional calculus for any selfadjoint unbounded operators with nonnegative spectrum.

II.3. Compact Operators

In the sequel for any $r > 0$ we denote by $B(0, r)$ the (open) unit ball of $\mathcal{H}$ of radius r about the origin.

DEFINITION II.5. *An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be compact when $\overline{T(B(0,1)}$ is compact in $\mathcal{H}$.*

We denote by $\mathcal{K}$ the space of compact operators.

The following lemma will be useful to study compact operators.

PROPOSITION II.5. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

(i) *T is a compact operator.*
(ii) *For any bounded sequence $(\xi_n)_{n \geq 0} \subset \mathcal{H}$ there is a subsequence $(\xi_{n_k})_{k \geq 0}$ such that the sequence $(T\xi_{n_k})_{k \geq 0}$ converges in norm.*
(iii) *For any sequence $(\xi_n)_{n \geq 0} \subset \mathcal{H}$ converging weakly to 0 the sequence $(T\xi_n)_{n \geq 0}$ converges to 0 in norm.*
(iv) *There is an orthonormal basis $(\xi_n)_{n \geq 0}$ of $\mathcal{H}$ such that*

$$\lim_{N \to \infty} \|T_{|E_N^\perp}\| = 0,$$

where we have denoted by E_N the span of $\xi_0, \ldots, \xi_{N-1}$.

(v) *For any orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$,

$$\lim_{N\to\infty} \|T_{|E_N^\perp}\| = 0,$$

where we have denoted by E_N *the span of* $\xi_0, \ldots, \xi_{N-1}$.

PROOF. Taking into account that the implication (v) $\Rightarrow$ (iv) is immediate, to prove the proposition we only need to prove the implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (v).

• (ii) $\Rightarrow$ (i): Assume that T sastifies the condition (ii). Then for any sequence $(\xi_n)_{n\geq 0} \subset B(0,1)$ the sequence $(T\xi_n)_{n\geq 0}$ admits a convergent subsequence. By virtue of the Bolzano-Weierstrass criterion this proves that $\overline{T(B(0,1))}$ is compact, i.e., T is a compact operator. Thus (ii) implies (i).

• (iii) $\Rightarrow$ (ii): Suppose that T satisfies the condition (iii). Let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be a bounded sequence, i.e., there exists $r > 0$ such that $\xi_n \in B(0,r)$ for all n. By Alaoglu theorem (see [**Fo**, pp. 169-170]) the ball $B(0,r)$ is precompact with respect to the weak topology and, as $\mathcal{H}$ is separable, the weak topology is metrizable (see [**Fo**]), so by the Bolzano-Weierstrass criterion there is a subsequence $(\xi_{n_k})_{k\geq 0}$ converging weakly to some ξ in $\mathcal{H}$. Then $\xi_{n_k} - \xi$ converges weakly to 0, so by (iii) the sequence $T(\xi_{n_k} - \xi)$ converges to 0 in norm, i.e., $T\xi_{n_k} \to T\xi$ in norm. This shows that (iii) implies (ii).

• (iv) $\Rightarrow$ (iii): Suppose that T satisfies (iv). Thus there exists an orthonormal basis $(\xi_n)_{n\geq 0}$ of $\mathcal{H}$ such that if, for any $N \in \mathbb{N}$, we denote by E_N the span of $\xi_0, \ldots, \xi_{N-1}$ then $\|T_{|E_N^\perp} \to 0$ as $N \to \infty$. In addition, we denote by Π_N the orthogonal projection onto E_N.

Let $(\eta_k)_{k\geq 0} \subset \mathcal{H}$ be a sequence converging weakly to 0. In particular $(\eta_k)_{k\geq 0}$ is weakly bounded, and hence is bounded in norm by the uniform boundedness principle and the fact $\mathcal{H}$ is isometrically isomorphic to its dual (see [**Fo**, pp. 163, 174–175]). Thus there exists $C > 0$ such that $\|\eta_k\| \leq C$ for all $k \in \mathbb{N}_0$.

Let $\epsilon > 0$. Since $\Pi_N \eta_k = \sum_{n<N} \langle \xi_n, \eta_k\rangle \xi_n$ we have

(II.3.1)
$$\begin{aligned}\|T\eta_k\| \leq \|T\Pi_N\eta_k\| + \|T(1-\Pi_N)\eta_k\| &\leq \sum_{n<N} |\langle \xi_n, \eta_k\rangle| \|T\xi_n\| + \|T(1-\Pi_N)\| \|\eta_k\| \\ &\leq \|T\| \sum_{n<N} |\langle \xi_n, \eta_k\rangle| + C\|T_{|E_N^\perp}\|.\end{aligned}$$

Since $\|T_{|E_N^\perp}\| \to 0$ as $N \to \infty$ by choosing N large enough we have

$$\|T(1-\Pi_N)\| < \epsilon. \tag{II.3.2}$$

As η_k converges weakly to 0 we see that $\sum_{n<N} |\langle \xi_n, \eta_k\rangle|$ goes to 0 as $k \to \infty$, and hence there exists $k_0 \in \mathbb{N}_0$ such that, for any $k \geq k_0$,

$$\sum_{n<N} |\langle \xi_n, \eta_k\rangle| < \epsilon. \tag{II.3.3}$$

Combining (II.3.1), (II.3.2) and (II.3.3) we see that, for all $k \geq k_0$, we have

$$\|T\eta_k\| \leq (\nu_0 + C)\epsilon.$$

This shows that $T\eta_k \to 0$ in norm as $k \to \infty$. Thus (iv) implies (iii).

• (i) ⇒ (v): Suppose that T is a compact operator. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For any $N \in \mathbb{N}$ we denote by E_N the span of $\xi_0, \ldots, \xi_{N-1}$. Assume that the sequence $(\|T_{|E_N^\perp}\|)_{N\geq 1}$ does not converge to 0 as $N \to 0$. Since this is a non-increasing sequence of non-negative numbers there is $c > 0$ such that $\|T_{|E_N^\perp}\| > c$ for all $N \in \mathbb{N}$. Therefore, for every $N \in \mathbb{N}$, there is a unit vector $\eta_N \in E_N^\perp$ such that $\|T\eta_N\| > c$.

Let $\xi \in \mathcal{H}$. As η_N is contained in $E_N^\perp$, and hence $\eta_N = \sum_{n\geq N}\langle \xi_n, \eta_N\rangle \xi_n$, we have

$$|\langle \eta_n, \xi\rangle| \leq \sum_{n\geq N} |\langle \eta_N, \xi_n\rangle\langle \xi, \xi_n\rangle| \leq \left(\sum_{n\geq N} |\langle \eta_N, \xi_n\rangle|^2\right)^2 \left(\sum_{n\geq N} |\langle \xi, \xi_n\rangle|^2\right)^2.$$

Since $\sum_{n\geq N} |\langle \eta_N, \xi_n\rangle|^2 = \|\eta_N\|^2 = 1$ and $\sum_{n\geq N} |\langle \xi, \xi_n\rangle|^2 \to 0$ as $N \to \infty$, it follows that $\langle \eta_n, \xi\rangle \to 0$ as $N \to \infty$. Thus η_N converges weakly to 0 as $N \to \infty$. As T is continuous with respect to the weak topology it follows that $T\eta_N$ converges weakly to 0 as $N \to \infty$.

On the other hand, the sequence $(T\eta_N)_{n\geq 1}$ is contained in the image by T of the unit sphere, which is precompact since T is compact. Therefore, by the Bolzano-Weierstrass criterion there is a subsequence $(\eta_{N_k})_{k\geq 0}$ such that $T\eta_{N_k}$ converges in norm to some ζ as $k \to \infty$. As $T\eta_{N_k}$ converges weakly to 0 we must have $\zeta = 0$, i.e., $T\eta_{N_k}$ converges to 0 in norm. This contradicts the fact that $\|T\eta_N\| > c$ for all $N \in \mathbb{N}$. Therefore, it is not possible for the sequence $(\|T_{|E_N^\perp}\|)_{N\geq 1}$ to not converge to 0. This proves that $\|T_{|E_N^\perp}\| \to 0$ as $N \to \infty$. Thus (i) implies (v). The proof is complete. □

Proposition II.6. *$\mathcal{K}$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

Proof. Let $(\xi_n)_{n\geq 0}$ be any sequence converging weakly to 0.

For $j = 1, 2$ let $T_j \in \mathcal{K}$ and $\lambda_j \in \mathbb{C}$. By Proposition II.5 (iii) the sequences $(T_1\xi_n)_{n\geq 0}$ and $(T_2\xi_n)_N$ converge to 0 in norm, and so $(\lambda_1 T_1 + \lambda_2 T_2)\xi_n)_{n\geq 0}$ too converges to 0 in norm. It then follows from Proposition II.5 (iii) that $\lambda_1 T_1 + \lambda_2 T_2$ is a compact operator. Thus $\mathcal{K}$ is a subspace of $\mathcal{L}(\mathcal{H})$.

Let $T \in \mathcal{K}$ and let $A, B \in \mathcal{L}(\mathcal{H})$. Since B is continuous with respect to the weak topology the sequence $(B_{\xi_n})_{n\geq 0}$ converges weakly to 0. Since T is compact Proposition II.5 (iii) insures us that $(TB\xi_n)_{n\geq 0}$ converges to 0 in norm. Then $(ATB\xi_n)_{n\geq 0}$ converges to 0 in norm too. Thanks to Proposition II.5 (iii), this shows that ATB is a compact operator. Therefore, we see that $\mathcal{K}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.

It remains to show that $\mathcal{K}$ is closed. Thus, let $(T_k)_{k\geq 0} \subset \mathcal{K}$ be a sequence such that $T_k \to T$ in $\mathcal{L}(\mathcal{H})$ and let us show that T is compact. Let $\epsilon > 0$. Then for k large enough $\|T - T_k\| < \epsilon$. Since T_k is compact by Proposition II.5 (iii) the sequence $(T_k\xi_n)_{n\geq 0}$ converges to 0 in norm, and hence there exists $N \in \mathbb{N}$, such that $\|T_k\xi_n\| < \epsilon$ for all $n \geq N$. Then, for any $n \geq N$, we have

$$\|T\xi_n\| \leq \|T_k\xi_n\| + \|(T - T_k)\xi_n\| \leq \|T_k\xi_n\| + \|T - T_k\|\|\xi_n\| < 2\epsilon.$$

Thus $(T\xi_n)_{n\geq 0}$ converges to 0 in norm, which by Proposition II.5 (iii) shows that T is compact. This proves that $\mathcal{K}$ is closed, completing the proof. □

Proposition II.7. *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$T \in \mathcal{K} \Longleftrightarrow T^* \in \mathcal{K} \Longleftrightarrow |T| \in \mathcal{K}. \qquad \text{(II.3.4)}$$

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ and let $T = U|T|$ be its polar decomposition. The fact that $\mathcal{K}$ is a two-sided ideal then implies that if $|T|$ is compact, then so is T. Likewise, as by Proposition II.3 we have $|T| = U^*T$, if T is compact, then so is $|T|$.

Proposition II.3 also tells us that $T^* = UTU^*$. Therefore, we also see that if T is compact then so is T^*. Upon substituting T^* for T, we see that if T^* is compact, then T is compact too. The proof is complete. □

Proposition II.8. *$\mathcal{K}$ is a sub-C^*-algebra of $\mathcal{L}(\mathcal{H})$, and hence is a C^*-algebra.*

Proof. Since Proposition II.6 tells us that $\mathcal{K}$ is a closed two-sided ideal, we see that $\mathcal{K}$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$. As by Proposition II.7 $\mathcal{K}$ is closed under the involution of $\mathcal{L}(\mathcal{H})$ it follows that $\mathcal{K}$ is a sub-C^*-algebra of $\mathcal{L}(\mathcal{H})$. □

Theorem II.5 (Riesz-Schauder; see [**RS**, Thm. VI.15]). *Let $T \in \mathcal{K}$.*

1. *T always contains 0 in its spectrum.*
2. *If $\lambda \in \operatorname{Sp} T \setminus 0$, then λ is an eigenvalue with finite multiplicity.*
3. *$\operatorname{Sp} T$ is either finite or consists of a sequence of complex numbers converging to 0.*

In the sequel, for any vectors ξ and η in $\mathcal{H}$, we denote by $\xi \otimes \eta^*$ the element of $\mathcal{L}(\mathcal{H})$ defined by

$$(\xi \otimes \eta^*)\zeta := \langle \eta, \xi' \rangle \xi \qquad \forall \zeta \in \mathcal{H}. \tag{II.3.5}$$

Thus in ketbra notation $\xi \otimes \eta^*$ is just the operator $|\xi\rangle\langle\eta|$. This is an operator of rank 1. If ξ is a unit vector then $\xi \otimes \xi^*$ is the orthogonal projection onto $\mathbb{C}\xi$.

Theorem II.6 (Hilbert-Schmidt; see [**RS**, Thm. VI.16]). *Let $T \in \mathcal{K}$ be normal. Then T diagonalizes in an orthonormal basis, i.e., there exists an orthonormal basis $(\xi_n)_{n \geq 0}$ of $\mathcal{H}$ and a sequence $(\lambda_n)_{n\geq 0} \subset \mathbb{C}$ such that*

$$T\lambda_n = \lambda_n \xi_n \qquad \forall n \in \mathbb{N}_0. \tag{II.3.6}$$

This result allows us to reinterpret the Borel functional calculus for normal compact operators as follows.

Proposition II.9. *Let $T \in \mathcal{K}$ be normal and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ respect to which T is diagonal, i.e., $T\lambda_n = \lambda_n\xi_n$ for all $n \in \mathbb{N}_0$. In addition, let f be bounded function on $\operatorname{Sp} T = \{\lambda_n;\ n \in \mathbb{N}_0\}$.*

1. *We have*
$$f(T) = \sum_{n\geq 0} f(\lambda_n)(\xi_n \otimes \xi_n^*), \tag{II.3.7}$$
where the series converges strongly.
2. *$f(T)$ is a compact operator if and only if*
$$\lim_{n\to\infty} f(\lambda_n) = 0.$$
Furthermore, in this case the series (II.3.7) converges in norm.

Proof. For any $n \in \mathbb{N}_0$ we have $T\xi_n = \lambda_n\xi_n$. Since $(\xi_n)_{n\geq 0}$ is an orthonormal basis we deduce that $T^*\xi_n = \overline{\lambda_n}\xi_n$ for all $n \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$. For any polynomial $f(z) = \sum a_{jk} z^j \bar{z}^k$, we have

$$f(T) = \sum a_{jk} T^j (T^*)^k \xi_n = \sum a_{jk} \lambda_n^j (\overline{\lambda_n})^k \xi_n = f(\lambda_n)\xi_n.$$

It then follows from (I.5.4) that, for any $f \in C(\operatorname{Sp} T)$, we have

$$f(T)\xi_n = \lambda_n \xi_n \qquad \forall n \in \mathbb{N}_0. \tag{II.3.8}$$

Since $\operatorname{Sp} T = \{\lambda_n; n \in \mathbb{N}_0\}$ any function on $\operatorname{Sp} T$ is a Borel function. Moreover, by Theorem II.5 0 is always in $\operatorname{Sp} T$ and the non-zero eigenvalues of T have finite multiplicities, i.e., each non-zero eigenvalue λ appears at most finitely many times in the sequence $(\lambda_n)_{n\geq 0}$. Thus $\lambda_n \to 0$ as $n \to \infty$, and hence a function $f(\lambda)$ on $\operatorname{Sp} T$ is continuous iff $\lim_{n\to\infty} f(\lambda_n) = f(0)$. Let f be a bounded function on $\operatorname{Sp} T$ and, for $N \in \mathbb{N}$, let f_N be the function on $\operatorname{Sp} T$ defined by

$$f_N(\lambda) = \begin{cases} f(\lambda_n) & \text{if } \lambda = \lambda_n \text{ with } n < N, \\ f(0) & \text{if } \lambda = 0 \text{ or } \lambda \neq \lambda_n \text{ for all } n < N. \end{cases}$$

As $\lim_{n\to\infty} f_N(\lambda_n) = f(0) = f_N(0)$, we see that f_N is a continuous function on $\operatorname{Sp} T$, and hence it satisfies (II.3.8). Moreover, for all $n \in \mathbb{N}_0$,

$$|f_N(\lambda_n)| \leq \|f\|_\infty \qquad \text{and} \qquad \lim_{N\to\infty} f_N(\lambda_n) = f(\lambda_n).$$

Thus $(f_N)_{N\geq 1}$ is a bounded sequence in $L^\infty(\operatorname{Sp} T)$ which converges pointwise to f. It then follows from Theorem II.2 and (II.3.8) that, for all $n \in \mathbb{N}_0$,

$$f(T)\xi_n = \lim_{N\to\infty} f_N(T)\xi_n = \lim_{N\to\infty} f_N(\lambda_n)\xi_n = f(\lambda_n)\xi_n. \tag{II.3.9}$$

Let $\xi \in \mathcal{H}$. Since $\xi = \sum_{n\geq 0}\langle \xi_n, \xi\rangle \xi_n$ and $f(T)$ is continuous, using (II.3.9) we get

$$f(T)\xi = \sum_{n\geq 0}\langle \xi_n, \xi\rangle f(\xi_n) = \sum_{n\geq 0}\langle \xi_n, \xi\rangle f(\lambda_n)\xi_n = \sum_{n\geq 0} f(\lambda_n)(\xi_n \otimes \xi_n^*)\xi,$$

which proves (II.3.7).

Next, for $N \in \mathbb{N}$ let E_N the span of $\xi_0, \ldots, \xi_{N_1}$ and denote by Π_N be the orthogonal projection onto E_N. Then by Proposition II.5 (iv) the operator $f(T)$ is compact if and only if

$$\|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\| \longrightarrow 0 \qquad \text{as } N \longrightarrow 0. \tag{II.3.10}$$

Set $\nu_N = \sup_{n\geq N} |f(\lambda_n)|$. Since $f(T)\xi_n = f(\lambda_n)\xi_n$ for all $n \in \mathbb{N}_0$, we see that $\nu_N \leq \|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\|$. Conversely, let $\xi \in \mathcal{H}$. Since $(\xi_n)_{n\geq 0}$ is an orthonormal basis, from (II.3.7) we get

$$\|f(T)(1-\Pi_N)\xi\|^2 = \sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2 |f(\lambda_n)|^2 \leq \nu_N^2 \sum_{n\geq 0} |\langle \xi_n, \xi\rangle|^2 = \nu_N^2 \|\xi\|^2.$$

Thus $\|f(T)(1-\Pi_N)\| \leq \nu_N$. Thefore, we have

$$\|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\| = \sup_{n\geq N} |f(\lambda_n)|. \tag{II.3.11}$$

Combining this with the condition (II.3.10) then shows that $f(T)$ is compact if and only if $\lim_{n\to\infty} f(\lambda_n) = 0$.

Finally, using (II.3.7) we get

$$f(T)(1-\Pi_N) = \sum_{n\geq N} f(\lambda_n)(\xi_n \otimes \xi_n^*),$$

where the series converges strongly. Thus,

$$\Big\| \sum_{n\geq N} f(\lambda_n)(\xi_n \otimes \xi_n^*) \Big\| = \|f(T)(1-\Pi_N)\| = \sup_{n\geq N} |f(\lambda_n)|.$$

Therefore, if $\lim_{n\to\infty} f(\lambda_n) = 0$, then the series (II.3.7) converges to $f(T)$ in norm. The proof is complete. $\square$

EXAMPLE II.6. Let $T \in \mathcal{K}$ be normal. Then, with the notation of Proposition II.9, we have

$$T = \sum_{\lambda_n \neq 0} \lambda_n (\xi_n \otimes \xi_n^*), \tag{II.3.12}$$

where the series converges in norm. Moreover, if we let $T = U|T|$ be the polar decomposition, then

$$U = \sum_{\lambda_n \neq 0} |\lambda_n|^{-1} \lambda_n (\xi_n \otimes \xi_n^*) \qquad \text{and} \qquad |T| = \sum_{n \geq 0} |\lambda_n| (\xi_n \otimes \xi_n^*),$$

where the first series converges strongly (unless T has finite rank, in which case it is a finite sum), and the second series converges in norm.

Bibliography

[Fo] Folland, G.: *Real analysis.* 2nd edition. John Wiley & Sons, Inc., New York, 1999.

[RS] Reed, M.; Simon, B.: *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, Inc., New York, 1980.

CHAPTER III

Characteristic Values and Operator Ideals

In this chapter we present basic results on characterstic values and ideals of compact operators. Part of the material of this chapter is succinctly presented in [**Co**, pp. 439–443] and [**GVF**, 310–317]. A thorough account on ideals of compact operators is given in the book of Gohberg-Krein [**GK**] (see also [**Si**]).

Throughout this chapter we let $\mathcal{H}$ be a separable Hilbert space.

III.1. Characteristic Values

Definition III.1 (Characteristic Values). *Let $T \in \mathcal{L}(\mathcal{H})$. For every $n \in \mathbb{N}_0$ the $(n+1)$-th characteristic value of T is*

$$\mu_n(T) := \inf\{\|T_{|E^\perp}\|;\ \dim E = n\}.$$

It immediately follows from this definition that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(\lambda T) = |\lambda| \mu_n(T) \qquad \forall \lambda \in \mathbb{C}\ \forall n \in \mathbb{N}_0. \tag{III.1.1}$$

In the sequel we denote by $\mathcal{R}_n$ the space of operators $T \in \mathcal{L}(\mathcal{H})$ of rank $\leq n$.

Proposition III.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n) \qquad \forall n \in \mathbb{N}_0, \tag{III.1.2}$$

$$\mu_m(T) \leq \mu_n(T) \qquad \forall m, n \in \mathbb{N}_0,\ m \geq n. \tag{III.1.3}$$

Proof. Let $n \in \mathbb{N}_0$. By definition $\operatorname{dist}(T, \mathcal{R}_n) = \inf\{\|T - R\|;\ R \in \mathcal{R}_n\}$. Let E be an n-dimensional subspace of $\mathcal{H}$ and denote by Π_E the orthogonal projection onto E. Then $\operatorname{rk} T\Pi_E \leq \operatorname{rk} \Pi_E = n$, i.e., $T\Pi_E$ is contained in $\mathcal{R}_n$. Thus,

$$\operatorname{dist}(T, \mathcal{R}_n) \leq \|T - T\Pi_E\| = \|T(1 - \Pi_E)\| = \|T_{|E^\perp}\|,$$

from which we deduce that $\operatorname{dist}(T, \mathcal{R}_n) \leq \mu_n(T)$.

Conversely, let $R \in \mathcal{R}_n$. As $\operatorname{rk} R^* = \operatorname{rk} R \leq n$ there exists an n-dimensional subspace E of $\mathcal{H}$ containing $\operatorname{im} R^*$, so that $E^\perp$ is contained in $(\operatorname{im} R^*)^\perp = \ker R$. Thus,

$$\|T - R\| \geq \sup_{\substack{\xi \in \ker R \\ \|\xi\|=1}} \|(T - R)\xi\| = \sup_{\substack{\xi \in \ker R \\ \|\xi\|=1}} \|T\xi\| \geq \sup_{\substack{\xi \in E^\perp \\ \|\xi\|=1}} \|T\xi\| = \|T_{|E^\perp}\| \geq \mu_n(T).$$

This implies that $\operatorname{dist}(T, \mathcal{R}_n) \geq \mu_n(T)$, and hence $\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n)$.

Next, let $m \in \mathbb{N}_0$, $m \geq n$. Then $\mathcal{R}_m \supset \mathcal{R}_n$, and so $\operatorname{dist}(T, \mathcal{R}_m) \leq \operatorname{dist}(T, \mathcal{R}_n)$. Since $\mu_m(T) = \operatorname{dist}(T, \mathcal{R}_m)$ and $\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n)$, the inequality (III.3) follows. The proof is complete. □

Proposition III.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $n \in \mathbb{N}_0$. Then*

$$\mu_n(T) = \mu_n(T^*) = \mu_n(|T|), \tag{III.1.4}$$

$$\mu_n(ATB) \leq \|A\|\mu_n(T)\|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}), \tag{III.1.5}$$

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary.} \tag{III.1.6}$$

Proof. Let A and B be in $\mathcal{L}(\mathcal{H})$, and let $R \in \mathcal{R}_n$. Then ARB is contained in $\mathcal{R}_n$ too, and hence

$$\mu_n(ATB) = \operatorname{dist}(ATB, \mathcal{R}_n) \leq \|ATB - ARB\| = \|A(T-R)B\| \leq \|A\|\|T-R\|\|B\|.$$

Thus $\mu_n(ATB) \leq \|A\| \operatorname{dist}(T, \mathcal{R}_n)\|B\| = \|A\|\mu_n(T)\|B\|$, proving (III.5).

If U is unitary and we take $A = U^*$ and $B = U$ in (III.5), then we get

$$\mu_n(U^*TU) \leq \|U^*\|\mu_n(T)\|U\| = \mu_n(T).$$

The above inequalities for U^*TU and U^* yield $\mu_n(T) \leq \mu_n(U^*TU)$. Thus $\mu_n(U^*TU)$ agrees with $\mu_n(T)$, proving (III.6).

It remains to prove (III.4). Let $T = U|T|$ be the polar decomposition of T. As $\|U\| = 1$ from (III.5) we get $\mu_n(T) = \mu_n(U|T|) \leq \|U\|\mu_n(|T|) = \mu_n(|T|)$. Since $|T| = U^*T$ we similarly have $\mu_n(|T|) \leq \mu_n(T)$, and hence $\mu_n(T) = \mu_n(|T|)$.

Finally, as $|T^*| = U^*|T|U$ and $|T| = U|T^*|U^*$, arguing as above shows that $\mu_n(|T|) = \mu_n(|T^*|)$. Since $\mu_n(|T^*|) = \mu_n(T^*)$ this implies that $\mu_n(|T|) = \mu_n(T^*)$, completing the proof. □

Remark III.2. Let $\mathcal{H}'$ be a separable Hilbert space, let $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and let $B \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$. Then by arguing as above we can show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(ATB) \leq \|A\|_{\mathcal{L}(\mathcal{H}',\mathcal{H})}\mu_n(T)\|B\|_{\mathcal{L}(\mathcal{H},\mathcal{H}')} \qquad \forall n \in \mathbb{N}_0.$$

In particular, if $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is unitary, then we can show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall n \in \mathbb{N}_0.$$

Thus the characteristic values of T are invariant under the action of unitary isomorphisms.

Proposition III.3. *Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $m, n \in \mathbb{N}_0$. Then*

$$\mu_{m+n}(S+T) \leq \mu_m(S) + \mu_n(T), \tag{III.1.7}$$

$$|\mu_m(S) - \mu_n(T)| \leq \|S - T\|, \tag{III.1.8}$$

$$\mu_{m+n}(ST) \leq \mu_m(S)\mu_n(T). \tag{III.1.9}$$

Proof. Let $R \in \mathcal{R}_m$ and let $R' \in \mathcal{R}_n$. Then $R + R'$ is contained in $\mathcal{R}_{m+m}$, and hence

$$\mu_{m+n}(S+T) = \operatorname{dist}(S+T, \mathcal{R}_{m+n}) \leq \|S + T - (R + R')\| \leq \|S - R\| + \|T - R\|.$$

Thus $\mu_{m+n}(S+T) \leq \operatorname{dist}(S, \mathcal{R}_m) + \operatorname{dist}(T, \mathcal{R}_n) = \mu_m(S) + \mu_n(T)$, proving (III.7).

If in (III.7) we substitute $S - T$ for S and we take $m = 0$, then we obtain

$$\mu_n(S) \leq \mu_0(S-T) + \mu_n(T) = \|S - T\| + \mu_n(T),$$

that is, $\mu_n(S) - \mu_n(T) \leq \|S - T\|$. Interchanging S and T yields

$$\mu_n(T) - \mu_n(S) \leq \|S - T\|,$$

so we see that $|\mu_m(S) - \mu_n(T)| \leq \|S - T\|$, i.e., (III.8) holds true.

It remains to prove (III.9). To this end let $R \in \mathcal{R}_m$ and let $R' \in \mathcal{R}_n$. Then

$$(S-R)(T-R') = ST - R'', \qquad R'' := SR' + R(T-R').$$

Observe that SR' is contained in $\mathcal{R}_n$ and $R(T-R')$ is contained in $\mathcal{R}_m$, so R'' is contained in $\mathcal{R}_{m+n}$. Thus,

$$\mu_{n+m}(ST) \leq \|ST - R''\| = \|(S-R)(T-R')\| \leq \|S-R\|\|T-R'\|.$$

From this we deduce that $\mu_{m+n}(ST) \leq \operatorname{dist}(S, \mathcal{R}_m)\operatorname{dist}(T, \mathcal{R}_n) = \mu_m(S)\mu_n(T)$, completing the proof. □

Proposition III.4. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

(i) *T is a compact operator.*
(ii) *T is the norm-limit of finite-rank operators.*
(iii) *$\mu_n(T) \to 0$ as $n \to 0$.*
(iv) *For any $\epsilon > 0$ there exists a finite-dimensional subspace E of $\mathcal{H}$ such that $\|T_{|E^\perp}\| < \epsilon$.*

Proof. The implication (i) $\Rightarrow$ (iv) follows from Proposition II.5 (iv)–(v). Therefore, to show that the conditions (i)–(iv) are equivalent it is enough to prove the implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii).

• (ii) $\Rightarrow$ (i): As any finite rank operator is compact and $\mathcal{K}$ is closed, any norm-limit of finite rank operators remains in $\mathcal{K}$. Thus (ii) implies (i).

• (iii) $\Rightarrow$ (ii): Suppose that $\lim_{n\to\infty} \mu_n(T) = 0$. Let $k \in \mathbb{N}$. Then there exists $n_k \in \mathbb{N}_0$ such that $\operatorname{dist}(T, \mathcal{R}_{n_k}) = \mu_{n_k}(T) < k^{-1}$. Thus there exists $R_k \in \mathcal{R}_{n_k}$ such that $\|T - R_k\| \leq 2k^{-1}$. Then the sequence $(R_k)_{k\geq 1}$ converges to T in norm, so T is a norm-limit of finite rank operators. This proves that (iii) implies (ii).

• (iv) $\Rightarrow$ (iii): Assume that (iv) holds. Let $\epsilon > 0$. Then there exists a finite-dimensional subspace $E \subset \mathcal{H}$ such that $\|T_{|E^\perp}\| < \epsilon$. Set $n_0 = \dim E$. Then we have $\mu_{n_0}(T) \leq \|T_{|E^\perp}\| < \epsilon$. Since by (III.3) the sequence $(\mu_n(T))_{n\geq 0}$ is decreasing, we see that $\mu_n(T) < \epsilon$ for all $n \geq n_0$. This proves that $\mu_n(T) \to 0$ as $n \to 0$. Thus (iv) implies (iii). The proof is complete. □

Theorem III.1 (Min-Max Principle). *Let $T \in \mathcal{K}$. Then, for all $n \in \mathbb{N}_0$,*

(III.1.10) $\qquad \mu_n(T) = (n+1)$*-th eigenvalue of $|T|$ counted with multiplicity.*

Proof. Since $\mu_n(T) = \mu_n(|T|)$ it enough to prove the result for $|T|$, which allows us to assume T positive. For any $k \in \mathbb{N}_0$ let λ_k be the $(k+1)$-th eigenvalue of T counted with multiplicity and let $(\xi_k)_{k\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $T\xi_k = \lambda_k \xi_k$ for all $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}$ denote by E_k be the k-dimensional subspace of $\mathcal{H}$ spanned by the vectors $\xi_0, \ldots, \xi_{k-1}$. Then by (II.11) we have

$$\|T_{|E_n^\perp}\| = \sup_{k\geq n} \lambda_k = \lambda_n.$$

Thus, by the very definition of $\mu_n(T)$ we have

(III.1.11) $$\mu_n(T) \leq \|T_{|E_n^\perp}\| = \lambda_n.$$

Conversely, let E be an n-dimensional subspace of $\mathcal{H}$ and denote by Π the orthogonal projection onto E. Since $\Pi_{|E_{n+1}}$ maps the $(n+1)$-dimensional space E_{n+1} to the n-dimensional subspace E, it cannot be one-to-one. Therefore, there exists a unit vector ξ which is contained in E_{n+1} and in $\ker \Pi = E^{\perp}$. In particular $\xi = \sum_{k\leq n} \alpha_k \xi_k$ with $\sum_{k\leq n} |\alpha_k|^2 = 1$. Thus,

$$\|T_{|E^{\perp}}\|^2 \geq \|T\xi\|^2 = \left\| \sum_{k\leq n} \alpha_k T\xi_k \right\|^2 = \left\| \sum_{k\leq n} \alpha_k \lambda_k \xi_k \right\|^2 = \sum_{k\leq n} |\alpha_k|^2 \lambda_k^2.$$

Since $\lambda_n \leq \lambda_k$ for all $k \leq n$, we deduce that

$$\|T_{|E^{\perp}}\|^2 \geq \lambda_n \sum_{k\leq n} |\alpha_k|^2 = \lambda_n.$$

As the inequality $\|T_{|E^{\perp}}\| \geq \lambda_n$ is valid for any n-dimensional subspace of $\mathcal{H}$, it follows that $\mu_n(T) \geq \lambda_n$. Together with (III.11) this shows that $\lambda_n = \mu_n(T)$. The proposition is thus proved. □

Let us now look at some interesting consequences of the min-max principle. First, in the light of (III.10) the inequality (III.8) shows the continuity of the eigenvalues of *positive* compact operators.

Another interesting consequence is the following.

Proposition III.5. *Let $T \in \mathcal{K}$ have polar decomposition $T = U|T|$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$.*

(1) *We have*

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converges in norm.

(2) *Let f be a non-negative non-decreasing function on $[0, \infty)$ which is continuous and vanishes at 0. Then*

$$\mu_n(f(|T|) = f(\mu_n(T)) \qquad \forall n \in \mathbb{N}_0, \tag{III.1.12}$$

$$f(|T|) = \sum_{n\geq 0} f(\mu_n(T))\xi_n \otimes \xi_n^*, \tag{III.1.13}$$

where the series converges in norm.

Proof. Let E be the closed subspace spanned by all the vectors ξ_n. Observe that E contains all the eigenvectors of $|T|$ associated to a non-zero eigenvalue. Therefore, E contains $(\ker |T|)^{\perp}$, and hence $E^{\perp}$ is contained in $\ker |T|$. Thus if we let $(\eta_k)_{k\in I}$ be an orthonormal basis of $E^{\perp}$, then $\{\eta_k\}_{k\in I} \cup \{\xi_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{H}$ which respect to which $|T|$ is diagonal, namely,

$$|T|\eta_k = 0 \quad \forall k \in I \qquad \text{and} \qquad |T|\xi_n = \mu_n(T)\xi_n \quad \forall n \in \mathbb{N}_0. \tag{III.1.14}$$

Then by II.12 we have

$$|T| = \sum_{k\in I} 0.\eta_k \otimes \eta_k^* + \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^* = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*, \tag{III.1.15}$$

where the series converge in norm. Since $T = U|T|$ this gives

$$T = U \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^* = \sum_{n\geq 0} \mu_n(T) U\left(\xi_n \otimes \xi_n^*\right) = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converge in norm.

Let f be a non-negative non-decreasing function on $[0, \infty)$ which is continuous and vanishes at 0. Then using (III.14) and Proposition II.9 we see that $f(|T|)$ is compact and we have

$$f(|T|) = \sum_{k \in I} f(0)\eta_k \otimes \eta_k^* + \sum_{n \geq 0} f(\mu_n(T))\xi_n \otimes \xi_n^* = \sum_{n \geq 0} f(\mu_n(T))\xi_n \otimes \xi_n^*,$$

where the series converges in norm. Since f is nondecreasing it follows from this that the $(n+1)$-th eigenvalue of $f(|T|)$ is $f(\mu_n(T))$. Therefore, using the min-max principle we deduce that

$$\mu_n(f(|T|) = f(\mu_n(T)) \qquad \forall n \in \mathbb{N}_0,$$

which completes the proof. □

EXAMPLE III.3. Let $p > 0$ and let $T \in \mathcal{K}$. Applying the above lemma to $f(t) = t^p$ (with the convention that $0^p = 0$) shows that

$$\mu_n(|T|^p) = \mu_n(T)^p \qquad \forall n \in \mathbb{N}_0.$$

III.2. Trace-class operators

For all $T \in \mathcal{L}(\mathcal{H})$ we set

$$\|T\|_1 := \sum_{n \geq 0} \mu_n(T).$$

We then define

$$\mathcal{L}^1 := \{T \in \mathcal{L}(\mathcal{H});\ \|T\|_1 < \infty\}. \tag{III.2.1}$$

The elements of $\mathcal{L}^1$ are called *trace-class operators.*

Observe that if $\sum \mu_n(T) < \infty$, then $\mu_n(T) \to 0$ as $n \to \infty$, and so using Proposition III.4 we see that T is a compact operator. Thus any trace-class operator is compact. Moreover,

$$\|T\| = \mu_0(T) \leq \|T\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H}). \tag{III.2.2}$$

Notice also that it follows from (III.1), (III.4) and (III.5) that

$$\|T\|_1 = \|T^*\|_1 = \||T|\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H}), \tag{III.2.3}$$

$$\|\lambda T\|_1 = |\lambda|\|T\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall \lambda \in \mathbb{C}, \tag{III.2.4}$$

$$\|ATB\|_1 \leq \|A\|\|T\|_1\|B\| \qquad \forall A, T, B \in \mathcal{L}(\mathcal{H}). \tag{III.2.5}$$

As an immediate consequence of (III.3) we see that, for all $T \in \mathcal{L}(\mathcal{H})$,

$$T \in \mathcal{L}^1 \Longrightarrow T^* \in \mathcal{L}^1 \Longrightarrow |T| \in \mathcal{L}^1.$$

In the sequel we denote by $\mathcal{L}(\mathcal{H})_+$ the cone of operators in $\mathcal{L}(\mathcal{H})$ that are positive.

LEMMA III.1. *Let $T \in \mathcal{L}(\mathcal{H})_+$. Then, for any orthonormal basis $(\xi_n)_{n \geq 0}$, we have*

$$\|T\|_1 = \sum_{n \geq 0} \langle \xi_n, T\xi_n \rangle. \tag{III.2.6}$$

Proof. Let us first assume that T is compact. By Theorem II.6 and Theorem III.1 there exists an orthonormal basis $(\eta_n)_{n\geq 0}$ such that $T\eta_n = \mu_n(T)\eta_n$ for all $n \in \mathbb{N}_0$. Then

$$\|T\|_1 = \sum_{n\geq 0} \mu_n(T) = \sum_{n\geq 0} \langle \eta_n, T\eta_n\rangle = \sum_{n\geq 0} \langle T^{\frac{1}{2}}\eta_n, T^{\frac{1}{2}}\eta_n\rangle = \sum_{n\geq 0} \|T^{\frac{1}{2}}\eta_n\|^2.$$

Since $(\xi_n)_{\geq 0}$ is an orthonormal basis, we also have

$$\begin{aligned}\sum_{n\geq 0} \|T^{\frac{1}{2}}\eta_n\|^2 = \sum_{n\geq 0}\Big(\sum_{k\geq 0} |\langle \xi_k, T^{\frac{1}{2}}\eta_n\rangle|^2\Big) &= \sum_{k\geq 0}\Big(\sum_{n\geq 0} |\langle T^{\frac{1}{2}}\xi_k, \eta_n\rangle|^2\Big)\\ &= \sum_{k\geq 0} \|T^{\frac{1}{2}}\xi_n\|^2 = \sum_{k\geq 0} \langle \xi_k, T\xi_k\rangle,\end{aligned}$$

proving the lemma when T is compact.

Suppose now that T is not compact. Then T is not trace-class, and hence $\|T\|_1 = \infty$. For $N \in \mathbb{N}$ denote by E_N the span of $\xi_0, \ldots, \xi_{N-1}$. Moreover, since T is positive we have $T = (T^{\frac{1}{2}})(T^{\frac{1}{2}})$, and so as $\mathcal{K}$ is a two-sided ideal $T^{\frac{1}{2}}$ cannot be compact. It then follows from Proposition II.5-(v) that the sequence $(\|(T^{\frac{1}{2}})_{|E_N^\perp}\|)_{N\geq 1}$ does not converge to 0. Since it is a non-increasing sequence of non-negative numbers this means there is $c > 0$ such that $\|(T^{\frac{1}{2}})_{|E_N^\perp}\| > c$ for all $N \in \mathbb{N}$.

Let $N \in \mathbb{N}$ and let $\xi \in E_N^\perp$ be such that $\|\xi\| = 1$ and $\|T^{\frac{1}{2}}\xi\| > c$. Notice that

$$\|T^{\frac{1}{2}}\xi\| = \Big\|\sum_{n\geq N} \langle \xi_n, \xi\rangle T^{\frac{1}{2}}\xi_n\Big\| \leq \sum_{n\geq N} |\langle \xi_n, \xi\rangle| \|T^{\frac{1}{2}}\xi_n\| \leq \Big(\sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\geq N} \|T^{\frac{1}{2}}\xi_n\|^2\Big)^{\frac{1}{2}}.$$

Since $\sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2 = \|\xi\|^2 = 1$ and $\|T^{\frac{1}{2}}\xi_n\|^2 = \langle T^{\frac{1}{2}}\xi_n, T^{\frac{1}{2}}\xi_n\rangle = \langle \xi_n, T\xi_n\rangle$, it follows that

$$c^2 < \|T\xi\|^2 \leq \sum_{n\geq N} \langle \xi_n, T\xi_n\rangle \qquad \forall N \in \mathbb{N}.$$

Therefore, the series $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ diverges, i.e., $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle = \infty = \|T\|_1$. The proof is complete. □

Definition III.4. *Let $T \in \mathcal{L}(\mathcal{H})_+$. Then the trace of T is defined to be*

$$\operatorname{Trace} T := \sum_{n\geq 0} \langle \xi_n, T\xi_n\rangle, \tag{III.2.7}$$

where $(\xi_n)_{n\geq 0}$ is any orthonormal basis.

Using (III.4) we see that, for all $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\|_1 = \||T|\|_1 = \operatorname{Trace}|T|.$$

In particular,

$$T \in \mathcal{L}^1 \Longleftrightarrow \operatorname{Trace}|T| < \infty.$$

We shall now extend the definition of $\operatorname{Trace} T$ to all operators $T \in \mathcal{L}^1$.

Lemma III.2. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$,*

$$\sum_{n\geq 0} |\langle \xi_n, T\xi_n\rangle| \leq \|T\|_1.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Then

$$|\langle \xi_n, T\xi_n\rangle| = |\langle \xi_n, U|T|\xi_n\rangle| = |\langle |T|^{\frac{1}{2}}U^*\xi_n, |T|^{\frac{1}{2}}\xi_n\rangle \leq \||T|^{\frac{1}{2}}U^*\xi_n\|\||T|^{\frac{1}{2}}\xi_n\|.$$

Thus,

$$\begin{aligned}\sum_{n\geq 0} |\langle \xi_n, T\xi_n\rangle| &\leq \sum_{n\geq 0} \||T|^{\frac{1}{2}}U^*\xi_n\|\||T|^{\frac{1}{2}}\xi_n\| \\ &\leq \left(\sum_{n\geq 0} \||T|^{\frac{1}{2}}U^*\xi_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 0} \||T|^{\frac{1}{2}}\xi_n\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{III.2.8}$$

Using Lemma III.1 and (III.2) we get

$$\sum_{n\geq 0} \||T|^{\frac{1}{2}}\xi_n\|^2 = \sum_{n\geq 0}\langle |T|^{\frac{1}{2}}\xi_n, |T|^{\frac{1}{2}}\xi_n\rangle = \sum_{n\geq 0}\langle \xi_n, |T|\xi_n\rangle = \operatorname{Trace}|T| = \|T\|_1.$$

Similarly, we have

$$\sum_{n\geq 0} \||T|^{\frac{1}{2}}U^*\xi_n\|^2 = \sum_{n\geq 0}\langle |T|^{\frac{1}{2}}U^*\xi_n, |T|^{\frac{1}{2}}U^*\xi_n\rangle = \sum_{n\geq 0}\langle \xi_n, U|T|U^*\xi_n\rangle.$$

Proposition II.3) tells us that $U^*|T|U = |T^*|$, and so using Lemma III.1 and (III.2) we get

$$\sum_{n\geq 0} \||T|^{\frac{1}{2}}U^*\xi_n\|^2 = \sum_{n\geq 0}\langle \xi_n, |T^*|\xi_n\rangle = \operatorname{Trace}|T^*| = \|T^*\|_1 = \|T\|_1. \tag{III.2.9}$$

Combining (III.8) with (III.2) and (III.9) gives

$$\sum_{n\geq 0} |\langle \xi_n, T\xi_n\rangle| \leq \|T\|_1^{\frac{1}{2}}\|T\|_1^{\frac{1}{2}} = \|T\|_1,$$

proving the lemma. □

Lemma III.3. *We have*

$$\|S+T\|_1 \leq \|S\|_1 + \|T\|_1 \qquad \forall S, T \in \mathcal{L}(\mathcal{H}). \tag{III.2.10}$$

Proof. Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $S+T = U|S+T|$ be the polar decompositions of $S+T$. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. Upon writing

$$|S+T| = U^*(S+T) = U^*S + U^*T$$

and using Lemma III.1 and Lemma III.2 we get

$$\begin{aligned}\|S+T\|_1 = \sum_{n\geq 0}\langle \xi_n, |S+T|\xi_n\rangle &\leq \sum_{n\geq 0} |\langle \xi_n, U^*S\xi_n\rangle| + \sum_{n\geq 0} |\langle \xi_n, U^*T\xi_n\rangle| \\ &\leq \|U^*S\|_1 + \|U^*T\|_1.\end{aligned}$$

Combining this with (III.5) gives

$$\|S+T\|_1 \leq \|U^*\|\|S\|_1 + \|U^*\|\|T\|_1 \leq \|S\|_1 + \|T\|_1,$$

proving the lemma. □

Proposition III.6. *The following hold.*

(1) *$\mathcal{L}^1$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
(2) *$\|.\|_1$ is a norm on $\mathcal{L}^1$ with respect to which $\mathcal{L}^1$ is a Banach space.*

Proof. It follows from (III.2), (III.4), (III.5) and (III.10) that $\mathcal{L}^1$ is a two-sided ideal and $\|.\|_1$ is a seminorm on $\mathcal{L}^1$.

It remains to show that $\mathcal{L}^1$ is complete with respect to the norm $\|.\|_1$. Thus, let $(T_n)_{n\geq 0}$ be a sequence with values in $\mathcal{L}^1$ which is Cauchy with respect to $\|.\|_1$. Since by (III.2) $\|T_n - T_p\| \leq \|T_p - T_n\|_1$ we see that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H})$, and hence converges to some operator T in $\mathcal{L}(\mathcal{H})$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|T_n - T_p\| < \epsilon$ for all $n, p \geq N$. By (III.8), for all $k \in \mathbb{N}$,

$$|\mu_k(T_n - T) - \mu_k(T_n - T_p)| \leq \|T_p - T\|.$$

Since $\|T_p - T\| \to 0$ as $p \to \infty$, we see that $\mu_k(T_n - T) = \lim_{p\to\infty} \mu_k(T_n - T_p)$. Therefore,

$$\sum_{k=0}^{M} \mu_k(T_n - T) = \lim_{p\to\infty} \sum_{k=0}^{M} \mu_k(T_n - T) \qquad \forall M \in \mathbb{N}.$$

Since for $n, p \geq N$ we have

$$\sum_{k=0}^{M} \mu_k(T_n - T) \leq \sum_{k=0}^{\infty} \mu_k(T_n - T) = \|T_n - T_p\|_1 < \epsilon,$$

we deduce that, if $n \geq N$, then $\sum_{k=0}^{M} \mu_k(T_n - T) \leq \epsilon$ for all $M \in \mathbb{N}$. Thus, for all $n \geq N$, we have

$$\|T_n - T\|_1 = \sum_{k=0}^{\infty} \mu_k(T_n - T) \leq \epsilon. \tag{III.2.11}$$

Therefore $T_n - T$ is in $\mathcal{L}^1$ and, as $\mathcal{L}^1$ is a subspace containing T_n, we see that T is trace-class. Then (III.11) shows that T_n converges to T with respect to the norm $\|.\|_1$. Thus $\mathcal{L}^1$ is complete with respect to the norm $\|.\|_1$, completing the proof. □

Remark III.5. It follows from (III.2) that the inclusions of $\mathcal{L}^1$ into $\mathcal{K}$ and $\mathcal{L}(\mathcal{H})$ are continuous.

Lemma III.4. *Any $T \in \mathcal{L}^1$ can be written in the form*

$$T = T_1 - T_2 + iT_3 - iT_4, \qquad T_j \in \mathcal{L}^1 \cap \mathcal{L}(\mathcal{H})_+.$$

Proof. Let $T \in \mathcal{L}^1$. Then $T = S_+ + iS_-$ with $S_\pm = \frac{1}{2}(T \pm T^*)$. The operators $S_\pm$ are selfadjoint and are contained in $\mathcal{L}^1$ by (III.2). Therefore, the proof reduces to proving that any selfadjoint trace-class operator can be written as the difference of two positive trace-class operators.

Let $T \in \mathcal{L}^1$ be selfadjoint. Then $T = T_+ - T_-$ with $T_\pm = \frac{1}{2}(|T| \pm T)$. Observe that by (III.2) the operators $T_\pm$ are trace-class. In addition, observe that $T_\pm = f_\pm(T)$ where $f_\pm(t) = \frac{1}{2}(|t| \pm t)$. As the functions $f_\pm$ are nonnegative on $\operatorname{Sp} T \subset \mathbb{R}$, the operators $T_\pm$ are positive by Corollary II.2. We deduce from this that T can be written as the difference of two positive trace-class operators. This completes the proof. □

We are now ready to prove the following.

Proposition III.7. *Let $T \in \mathcal{L}^1$. For any orthonormal basis $(\xi_n)_{n\geq 0}$ the series*

$$\sum_{n\geq 0} \langle \xi_n, T\xi_n \rangle$$

is absolutely convergent and the value of its sum does not depend on the choice of the orthonormal basis $(\xi_n)_{n\geq 0}$.

PROOF. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. It follows from Lemma III.2 that the series $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ is absolutely convergent. Moreover, Lemma III.4 allows us to write $T = T_1 - T_2 + iT_3 - iT_4$ with $T_j \in \mathcal{L}^1 \cap \mathcal{L}(\mathcal{H})_+$. Then using Lemma III.1 we get

$$\begin{aligned}\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle &= \sum_{n\geq 0}\langle \xi_n, T_1\xi_n\rangle - \sum_{n\geq 0}\langle \xi_n, T_2\xi_n\rangle + i\sum_{n\geq 0}\langle \xi_n, T_3\xi_n\rangle - i\sum_{n\geq 0}\langle \xi_n, T_4\xi_n\rangle \\ &= \operatorname{Trace} T_1 - \operatorname{Trace} T_2 + i\operatorname{Trace} T_3 - i\operatorname{Trace} T_4.\end{aligned}$$

Therefore the value of the sum $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ does not depend on the orthonormal basis $(\xi_n)_{n\geq 0}$, proving the proposition. □

DEFINITION III.6. *The trace of an operator* $T \in \mathcal{L}^1$ *is defined to be*

$$\operatorname{Trace}(T) := \sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle,$$

where $(\xi_n)_{n\geq 0}$ *is any orthonormal basis of* $\mathcal{H}$.

LEMMA III.5. *Any* $A \in \mathcal{L}(\mathcal{H})$ *is linear combination of* 4 *unitaries.*

PROOF. Upon writing $A = \|A\|(T_+ + iT_-)$ with $T_\pm = \frac{1}{2\|A\|}(T \pm T^*)$ we see that the proof reduces to showing that any selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ with $\|A\| \leq 1$ is a linear combination of two unitaries.

Let $A \in \mathcal{L}(\mathcal{H})$ be selfadjoint and such that $\|A\| \leq 1$. Then the spectrum of A is contained in $[-1, 1]$, and so we can write

$$A = \frac{1}{2}(U + U^*), \qquad U := A + i\sqrt{1 - A^2}.$$

Observe that $U = f(A)$ with $f(t) = 1 + i\sqrt{1-t^2}$ and $f(t)\overline{f(t)} = 1$, so by continuous functional calculus $U^*U = UU^* = f(A)\bar{f}(A) = 1$, i.e., U is unitary. Thus A is the linear combination of two unitaries. This completes the proof. □

PROPOSITION III.8. *The map* $T \to \operatorname{Trace}(T)$ *is a linear form on* $\mathcal{L}^1$ *such that, for any* $T \in \mathcal{L}^1$, *we have*

$$|\operatorname{Trace}(T)| \leq \|T\|_1, \tag{III.2.12}$$

$$\operatorname{Trace}(T^*) = \overline{\operatorname{Trace}(T)}, \tag{III.2.13}$$

$$\operatorname{Trace}(AT) = \operatorname{Trace}(TA) \qquad \forall A \in \mathcal{L}(\mathcal{H}). \tag{III.2.14}$$

PROOF. It is immediate from its definition that the map $T \to \operatorname{Trace}(T)$ is linear. Moreover, if $T \in \mathcal{L}^1$, then by Lemma III.2, for any orthonormal basis $(\xi_n)_{n\geq 0}$,

$$|\operatorname{Trace}(T)| = |\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle| \leq \sum_{n\geq 0}|\langle \xi_n, T\xi_n\rangle| \leq \|T\|_1.$$

Moreover, by (III.2) the adjoint T^* is trace-class. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$, we have

$$\operatorname{Trace}(T^*) = \sum_{n\geq 0}\langle \xi_n, T^*\xi_n\rangle = \sum_{n\geq 0}\overline{\langle \xi_n, T\xi_n\rangle} = \overline{\operatorname{Trace}(T)}.$$

Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Then

$$\operatorname{Trace}(U^*TU) = \sum_{n\geq 0} \langle \xi_n, U^*TU\xi_n\rangle = \sum_{n\geq 0} \langle U\xi_n, TU\xi_n\rangle.$$

Since U is unitary $(U\xi_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$. Therefore, in view of the definition of $\operatorname{Trace}(T)$ we see that $\sum_{n\geq 0}\langle U\xi_n, TU\xi_n\rangle = \operatorname{Trace}(T)$. Thus $\operatorname{Trace}(U^*TU) = \operatorname{Trace}(T)$. Upon replacing T by UT we deduce that

$$\operatorname{Trace}(TU) = \operatorname{Trace}(UT) \qquad \forall U \in \mathcal{L}(\mathcal{H}) \text{ unitary}.$$

As by Lemma III.5 any $A \in \mathcal{L}(\mathcal{H})$ is the linear combination of 4 unitaries, it follows that

$$\operatorname{Trace}(AT) = \operatorname{Trace}(TA) \qquad \forall A \in \mathcal{L}(\mathcal{H}).$$

This shows that $T \rightarrow \operatorname{Trace}(T)$ is a trace on $\mathcal{L}^1$. The proof is complete. □

Example III.7. Let ξ and η be nonzero vectors in $\mathcal{H}$. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $\xi_0 = \|\xi\|^{-1}\xi$. Then, for any $n \in \mathbb{N}_0$,

$$\langle \xi_n, (\xi\otimes\eta^*)\xi_n\rangle = \langle \xi_n, \xi\rangle\langle \eta, \xi_n\rangle = \delta_{n,0}\|\xi\|^{-1}\langle \eta, \xi_0\rangle = \delta_{n,0}\langle \eta, \xi\rangle.$$

Therefore, we get

$$\operatorname{Trace}(\xi\otimes\eta^*) = \sum_{n\geq 0}\langle \xi_n, (\xi\otimes\eta^*)\xi_n\rangle = \langle \eta, \xi\rangle. \tag{III.2.15}$$

Example III.8. Let $T \in \mathcal{L}^1$ be normal. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ in which T diagonalizes, i.e., $T\lambda_n = \lambda_n\xi_n$ forall $n \in \mathbb{N}_0$. Then

$$\operatorname{Trace} T = \sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle = \sum_{n\geq 0}\langle \xi_n, \lambda_n\xi_n\rangle = \sum_{n\geq 0}\lambda_n, \tag{III.2.16}$$

that is, $\operatorname{Trace} T$ is the sum of the eigenvalues of T counted with multiplicity.

III.3. Duality between $\mathcal{L}^1$ and $\mathcal{L}(\mathcal{H})$

The Banach space $\mathcal{L}^1$ is in duality with the Banach spaces $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}$. This can be seen as follows.

If S and T are in $\mathcal{L}(\mathcal{H})$ and one of these operators is trace-class, we set

$$(S,T) := \operatorname{Trace}(ST).$$

Thanks to (III.5) and (III.12) we see that, if $S \in \mathcal{L}^1$ and $T \in \mathcal{L}(H)$, then

$$|(T,S)| = |(S,T)| = |\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_1\|T\|. \tag{III.3.1}$$

From this we deduce the following:

- For any $S \in \mathcal{L}(\mathcal{H})$ the map $(S,.) : \mathcal{L}^1 \ni T \rightarrow \operatorname{Trace}(ST)$ is a continuous linear form on $\mathcal{L}^1$.
- For any $S \in \mathcal{L}^1$ the map $(S,.) : \mathcal{L}(\mathcal{H}) \in T \rightarrow \operatorname{Trace}(ST)$ gives rise to a continuous linear form on $\mathcal{L}(\mathcal{H})$.

Lemma III.6. *For any ξ and η in $\mathcal{H}$, we have*

$$\|\xi\otimes\eta^*\| = \|\xi\otimes\eta^*\|_1 = \|\xi\|\|\eta\|.$$

Proof. First, in view of the definition (II.5) of $\xi \otimes \eta^*$ we have

$$\|\xi \otimes \eta^*\| = \sup_{\|\zeta\|=1} \|(\xi \otimes \eta^*)\zeta\| = \sup_{\|\zeta\|=1} \|\langle \eta, \zeta \rangle \xi\| = (\sup_{\|\zeta\|=1} |\langle \eta, \zeta \rangle|)\|\xi\| = \|\eta\|\|\xi\|.$$

Moreover, for any ζ_1 and ζ_2 in $\mathcal{H}$ we have

$$\langle \zeta_1, (\xi \otimes \eta^*)\zeta_2 \rangle = \langle \eta, \zeta_2 \rangle \langle \zeta_1, \xi \rangle = \langle \langle \xi, \zeta_1 \rangle \eta, \zeta_2 \rangle = \langle (\eta \otimes \xi^*)\zeta_1, \zeta_2 \rangle,$$

which shows that

$$(\xi \otimes \eta^*)^* = \eta \otimes \xi^*. \tag{III.3.2}$$

We then have

$$(\xi \otimes \eta^*)^*(\xi \otimes \eta^*) = (\eta \otimes \xi^*)(\xi \otimes \eta^*) = \|\xi\|^2(\eta \otimes \eta^*).$$

Notice that $\|\eta\|^{-2}(\eta \otimes \eta^*)$ is the orthogonal projection onto $\mathbb{C}\eta$, and hence, as any orthogonal projection, $\|\eta\|^{-2}(\eta \otimes \eta^*)$ is a positive operator which agrees with its square root, i.e., $\eta \otimes \eta^*$ is positive and $(\eta \otimes \eta^*)^{\frac{1}{2}} = \|\eta\|^{-1}(\eta \otimes \eta^*)$. Thus,

$$|\xi \otimes \eta^*| = ((\xi \otimes \eta^*)^*(\xi \otimes \eta^*))^{\frac{1}{2}} = (\|\xi\|^2(\eta \otimes \eta^*))^{\frac{1}{2}} = \|\xi\|\|\eta\|^{-1}(\eta \otimes \eta^*).$$

Combining this with (III.15) we obtain

$$\|\xi \otimes \eta^*\|_1 = \operatorname{Trace}|\xi \otimes \eta^*| = \operatorname{Trace}\left(\|\xi\|\|\eta\|^{-1}(\eta \otimes \eta)^*\right) = \|\xi\|\|\eta\|,$$

completing the proof. $\square$

Lemma III.7. *The following hold.*

(i) *For all $S \in \mathcal{L}^1$,*

$$\|S\|_1 = \sup_{\|T\|=1} |\operatorname{Trace}(ST)|. \tag{III.3.3}$$

(ii) *For all $S \in \mathcal{L}(\mathcal{H})$,*

$$\|S\| = \sup_{\|T\|_1=1} |\operatorname{Trace}(ST)|. \tag{III.3.4}$$

Proof. Let $S \in \mathcal{L}^1$. Then by (III.1), for any $T \in \mathcal{L}(\mathcal{H})$ of norm 1, we have

$$|\operatorname{Trace}(ST)| \leq \|S\|_1\|T\| = \|S\|_1. \tag{III.3.5}$$

Let $S = U|S|$ be the polar decomposition of S. Then by Proposition II.2 $\|U^*\| = \|U\| = 1$ and $U^*S = |S|$, and hence

$$\operatorname{Trace}(SU^*) = \operatorname{Trace}(U^*S) = \operatorname{Trace}|S| = \|S\|_1.$$

Together with (III.5) this gives (III.3).

Now, let $S \in \mathcal{L}(\mathcal{H})$. We may assume $S \neq 0$, since otherwise the equality (III.4) is trivially satisfied. By (III.1), for any T in the unit sphere of $\mathcal{L}^1$,

$$|\operatorname{Trace}(ST)| \leq \|S\|\|T\|_1 = \|S\|. \tag{III.3.6}$$

Moreover, as

$$\sup_{\|\xi\|=1} \langle \xi, |S|\xi \rangle = \sup_{\|\xi\|=1} \||S|^{\frac{1}{2}}\xi\|^2 = \||S|^{\frac{1}{2}}\|^2 = \|(S|^{\frac{1}{2}})^*S|^{\frac{1}{2}}\| = \||S|\| = \|S\|,$$

we see that, for any $\epsilon \in (0, \|S\|)$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\langle \xi, |S|\xi \rangle \geq \|S\| - \epsilon. \tag{III.3.7}$$

Let $S = U|S|$ be the polar decomposition of S. Since $\|U\| = 1$, we have $\|U\xi\| \leq \|\xi\| = 1$. Moreover, by Lemma II.1 and Proposition II.2 $\ker U = \ker S = \ker |S|$, and so (III.7) implies that $U\xi$ is not in $\ker U$. Therefore, using Lemma III.6 we see that

$$\|\xi \otimes (U\xi)^*\|_1 = \|\xi\|\|U\xi\| = \|U\xi\| \in (0,1].$$

In addition, using (III.15) and the fact that $|S| = U^*S$ we get

$$\text{Trace}\,[S(\xi \otimes (U\xi)^*)] = \text{Trace}\,((S\xi) \otimes (U\xi)^*) = \langle U\xi, S\xi\rangle = \langle \xi, U^*S\xi\rangle = \langle \xi, |S|\xi|\rangle.$$

Combining this with (III.7) then gives

$$\text{Trace}\,[S(\xi \otimes (U\xi)^*)] \geq \|S\| - \epsilon. \tag{III.3.8}$$

Set $T = c^{-1}(\xi \otimes (U\xi)^*)$, with $c = \|\xi \otimes (U\xi)^*\|_1$. Then $\|T\|_1 = 1$ and, as $c^{-1} \geq 1$, using (III.8) we see that, for any $\epsilon \in (0, \|S\|)$, we have

$$\text{Trace}\, ST = c^{-1}\,\text{Trace}\,[S(\xi \otimes (U\xi)^*)] \geq c^{-1}(\|S\| - \epsilon) \geq \|S\| - \epsilon.$$

Combining this with (III.6) yields (III.4). The proposition is thus proved. □

In the sequel we denote by $\mathcal{R}_\infty$ the subspace of $\mathcal{L}(\mathcal{H})$ consisting of finite rank operators. This is the subspace of $\mathcal{H}$ spanned by the rank 1, i.e., by all operators of the form (II.5).

We know that the closure of $\mathcal{R}_\infty$ in $\mathcal{L}(\mathcal{H})$ is $\mathcal{K}$ (cf. Proposition III.4). In addition, the following holds.

Lemma III.8. *The finite-rank operators are dense in $\mathcal{L}^1$.*

Proof. Let $T \in \mathcal{L}^1$ have polar decomposition $T = U|T|$, and let $(\xi_n)_{n\geq 0}$ be an orthonormal family of $\mathcal{H}$ such that $T\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Then by Proposition III.5 we have

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converges in norm. By Lemma III.6 we have

$$\|(U\xi_n) \otimes \xi_n^*\|_1 = \|U\xi_n\|\|\xi_n\| \leq \|U\|\|\xi_n\|^2 = 1,$$

and so we see that

$$\sum_{n\geq 0} \|\mu_n(T)(U\xi_n) \otimes \xi_n^*\|_1 \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1 < \infty.$$

Thus the series $\sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges in $\mathcal{L}^1$. Since it converges to T in $\mathcal{L}(\mathcal{H})$ and the inclusion of $\mathcal{L}^1$ into $\mathcal{L}(\mathcal{H})$ is continuous (cf. Remark III.5), T is its sum in $\mathcal{L}^1$ too, that is, T is the limit in $\mathcal{L}^1$ of the finite-rank operators $\sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^*$. This shows that finite-rank operators are dense in $\mathcal{L}^1$, proving the lemma. □

Proposition III.9. *The map $\mathcal{L}(\mathcal{H}) \ni S \to (S,.) \in (\mathcal{L}^1)'$ is an isometric isomorphism.*

Proof. It follows from (III.4) that the map $\mathcal{L}(\mathcal{H}) \ni S \to (S,.) \in (\mathcal{L}^1)'$ is isometric. Therefore, in view of Lemma I.1, in order to prove this map is an isometric isomorphism we only have to check it is onto.

Let $\varphi \in (\mathcal{L}^1)'$. Let S be the endomorphism of $\mathcal{H}$ defined by

$$\langle \eta, S\xi\rangle = \langle \varphi, \xi \otimes \eta^*\rangle \qquad \forall \xi, \eta \in \mathcal{H}.$$

Thanks to Lemma III.6 we have

$$|\langle \eta, S\xi\rangle| = |\langle \varphi, \xi \otimes \eta^*\rangle| \leq \|\varphi\|_{(\mathcal{L}^1)'}\|\langle \varphi, \xi \otimes \eta^*\rangle\|_1 \leq \|\varphi\|_{(\mathcal{L}^1)'}\|\|\xi\|\|\eta\|.$$

Thus,

$$\sup_{\|\xi\|=1} \|S\xi\| = \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} |\langle \eta, S\xi \rangle| \leq \|\varphi\|_{(\mathcal{L}^1)'},$$

showing that S is a continuous endomorphism, i.e., S is contained in $\mathcal{L}(\mathcal{H})$.

Now, thanks to III.15, for any ξ and η in $\mathcal{H}$, we have

$$(S, \xi \otimes \eta^*) = \text{Trace}(S(\xi \otimes \eta^*)) = \text{Trace}((S\xi) \otimes \eta^*) = \langle \eta, S\xi \rangle = \langle \varphi, \xi \otimes \eta^* \rangle, \tag{III.3.9}$$

that is, $(S, .)$ and φ agree on operators of rank 1, and by linearity they agree on their span, namely, $\mathcal{R}_\infty$. Both $(S, .)$ and φ are continuous linear forms and by Lemma III.8 the subspace $\mathcal{R}_\infty$ is dense in $\mathcal{L}^1$, so $(S, .)$ and φ agree on all $\mathcal{L}^1$. This proves that the map $\mathcal{L}(\mathcal{H}) \ni S \to (S, .) \in (\mathcal{L}^1)'$ is onto, completing the proof. □

It follows from the above the proposition that $\mathcal{L}(\mathcal{H})$ can be canonically identified with the dual of $\mathcal{L}^1$. For this reason $\mathcal{L}^1$ can be referred to as the *predual* of $\mathcal{L}(\mathcal{H})$.

The converse is not true. Namely, $\mathcal{L}^1$ is not the dual of $\mathcal{L}(\mathcal{H})$, but instead is that of the space $\mathcal{K}$ of compact operators. This is the content of the following.

Proposition III.10. *The map $\mathcal{L}^1 \ni S \to (S, .) \in \mathcal{K}'$ is an isometric isomorphism.*

Proof. As in the proof of Proposition III.9 we only have to prove that the map $\mathcal{L}^1 \ni S \to (S, .) \in \mathcal{K}'$ is onto. To this end let $\varphi \in \mathcal{K}'$. Since the inclusion of $\mathcal{L}^1$ into $\mathcal{K}$ is contionuous (cf. Remark III.5), we see that φ induces a continuous linear form on $\mathcal{L}^1$, and hence by Proposition III.9 there exists $S \in \mathcal{L}(\mathcal{H})$ such that

$$\langle \varphi, T \rangle = \text{Trace}(ST) \qquad \forall T \in \mathcal{L}^1. \tag{III.3.10}$$

In particular, by (III.9) we have

$$\langle \eta, S\xi \rangle = \langle \varphi, \xi \otimes \eta^* \rangle \qquad \forall \xi, \eta \in \mathcal{H}. \tag{III.3.11}$$

Let $S = U|S|$ be the polar decomposition of S and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. As by Proposition II.2 $|S| = U^*S$, using (III.11) we see that, for any $n \in \mathbb{N}_0$, we have

$$\langle \xi_n, |S|\xi_n \rangle = \langle \xi_n, U^*S\xi_n \rangle = \langle U\xi_n, S\xi_n \rangle = \langle \varphi, \xi_n \otimes (U\xi_n)^* \rangle. \tag{III.3.12}$$

Notice that thanks to (III.2) we have

$$\xi_n \otimes (U\xi_n)^* = ((U\xi_n) \otimes \xi_n^*)^* = (U(\xi_n \otimes \xi_n^*))^* = (\xi_n \otimes \xi_n^*)U^*,$$

and hence $\langle \xi_n, |S|\xi_n \rangle = \langle \varphi, (\xi_n \otimes \xi_n^*)U^* \rangle$. Therefore, for all $N \in \mathbb{N}$,

$$\begin{aligned} \sum_{n<N} \langle \xi_n, |S|\xi_n \rangle = \left| \langle \varphi, (\sum_{n<N} \xi_n \otimes \xi_n^*)U^* \rangle \right| &\leq \|\varphi\|_{\mathcal{K}'} \left\| (\sum_{n<N} \xi_n \otimes \xi_n^*)U^* \right\| \\ &\leq \|\varphi\|_{\mathcal{K}'} \left\| \sum_{n<N} \xi_n^* \otimes \xi_n \right\| \|U^*\|. \end{aligned}$$

By Proposition II.2 $\|U^*\| = \|U\| = 1$ and, as $\sum_{n<N} \xi_n \otimes \xi_n^*$ is the orthogonal projection onto the span of $\xi_0, \ldots, \xi_N$, this operator has norm 1 too. Thus,

$$\sum_{n<N} \langle \xi_n, |S|\xi_n \rangle \leq \|\varphi\|_{\mathcal{K}'} \qquad \forall N \in \mathbb{N}.$$

Using (III.3) and Lemma III.1 we then get

$$\|S\|_1 = \||S|\|_1 = \sum_{n\geq 0} \langle \xi_n, |S|\xi_n \rangle \leq \|\varphi\|_{\mathcal{K}'} < \infty,$$

and hence S is trace-class.

Since S is in $\mathcal{L}^1$, the map $(S,.) : \mathcal{K} \ni T \to \text{Trace}(ST)$ is a continuous linear map on $\mathcal{K}$. Moreover, it follows from (III.10) that it agrees with φ on $\mathcal{L}^1$, and hence on $\mathcal{R}_\infty$. As by Proposition III.4 $\mathcal{R}_\infty$ is dense in $\mathcal{K}$, we see that $(S,.)$ and φ agree on all $\mathcal{K}$, showing that the map $\mathcal{L}^1 \ni S \to (S,.) \in \mathcal{K}'$ is onto. The proof is complete. $\square$

III.4. Hilbert-Schmidt Operators

For any $T \in \mathcal{L}(\mathcal{H})$ we define

$$\|T\|_2 = \left(\sum_{n\geq 0} \mu_n(T)^2\right)^{\frac{1}{2}}.$$

We then define

$$\mathcal{L}^2 := \{T \in \mathcal{L}(\mathcal{H});\ \|T\|_2 < \infty\}.$$

The elements of $\mathcal{L}^2$ are called Hilbert-Schmidt operators.

As in (III.3)–(III.5) we have

$$\|T\|_2 = \|T^*\|_2 = \||T|\|_2 \qquad \forall T \in \mathcal{L}(\mathcal{H}), \tag{III.4.1}$$

$$\|\lambda T\|_2 = |\lambda|\|T\|_2 \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall \lambda \in \mathbb{C}, \tag{III.4.2}$$

$$\|ATB\|_2 \leq \|A\|\|T\|_2\|B\| \qquad \forall A, T, B \in \mathcal{L}(\mathcal{H}). \tag{III.4.3}$$

As an immediate consequence of (III.1) we see that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$T \in \mathcal{L}^2 \Longleftrightarrow T^* \in \mathcal{L}^2 \Longleftrightarrow |T| \in \mathcal{L}^2.$$

Lemma III.9. *The following hold.*

(i) *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$\|T\| \leq \|T\|_2 \leq \|T\|^{\frac{1}{2}}\|T\|_1^{\frac{1}{2}}. \tag{III.4.4}$$

(ii) *We have the inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K}. \tag{III.4.5}$$

Proof. Let $T \in \mathcal{L}(\mathcal{H})$. Since $\|T\| = \mu_0(T)$ we have $\|T\| \leq \|T\|_2$. Moreover, using (III.3) we see that, for any $n \in \mathbb{N}_0$,

$$\mu_n(T)^2 = \mu_n(T).\mu_n(T) \leq \mu_0(T)\mu_n(T) = \|T\|\mu_n(T).$$

Thus,

$$\sum_{n\geq 0} \mu_n(T)^2 \leq \|T\| \sum_{n\geq 0} \mu_n(T) = \|T\|\|T\|_1,$$

from which we get $\|T\|_2 \leq \|T\|^{\frac{1}{2}}\|T\|_1^{\frac{1}{2}}$. In particular, if $T \in \mathcal{L}^1$, then $\|T\|_2 < \infty$, and hence T is in $\mathcal{L}^2$. Thus $\mathcal{L}^1$ is contained in $\mathcal{L}^2$.

Let $T \in \mathcal{L}^2$. Then $\sum_{n\geq 0} \mu_n(T)^2 < \infty$, and hence $\lim_{n\to\infty} \mu_n(T) = 0$. Therefore T is compact by Proposition III.4. This proves that $\mathcal{L}^2$ is contained in $\mathcal{K}$. $\square$

Example III.9. Let ξ and η be vectors in $\mathcal{H}$. By Lemma III.6 both $\|\xi \otimes \eta^*\|$ and $\|\xi \otimes \eta^*\|_1$ are equal to $\|\xi\|\|\eta\|$, and so using (III.4) we get

$$\|\xi \otimes \eta^*\|_2 = \|\xi\|\|\eta\|.$$

LEMMA III.10. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$, we have*

$$\|T\|_2^2 = \operatorname{Trace}|T|^2 = \sum_{n\geq 0} \|T\xi_n\|^2, \tag{III.4.6}$$

and hence

$$T \in \mathcal{L}^2 \Longleftrightarrow \operatorname{Trace}|T|^2 < \infty \Longleftrightarrow |T|^2 \in \mathcal{L}^1. \tag{III.4.7}$$

PROOF. Assume first that T is compact. Then thanks to (III.12) $\mu_n(|T|^2) = \mu_n(T)^2$ for all $n \in \mathbb{N}_0$, and so we have

$$\|T\|_2 = \left(\sum_{n\geq 0} \mu_n(|T|^2\right)^{\frac{1}{2}} = (\operatorname{Trace}|T|^2)^{\frac{1}{2}}.$$

Suppose now that T is not compact. Then by (III.5) T is not in $\mathcal{L}^2$, and hence $\|T\|_2 = \infty$. By Proposition II.7 the fact that T is not compact, implies that $|T|$ is not compact either. Observe further that, as $\lim_{t\to 0^+} t^{\frac{1}{2}} = 0$, it follows from Proposition II.9 that, for any positive compact operator S, the operator $S^{\frac{1}{2}}$ is compact too. Therefore, if $|T|^2$ were compact, then $|T| = (|T|^2)^{\frac{1}{2}}$ would be compact too. Since $|T$ is not compact, we deduce that $|T|^2$ cannot be compact. Incidentally, $|T|^2$ is not trace-class, and hence

$$\operatorname{Trace}|T|^2 = \||T|^2\|_1 = \infty = \|T\|_2.$$

In general, for any $T \in \mathcal{L}(\mathcal{H})$, using Lemma III.2 we get

$$\|T\|_2^2 = \operatorname{Trace}|T|^2 = \sum_{n\geq 0} \langle \xi_n, |T|^2\xi_n\rangle. \tag{III.4.8}$$

Since $|T|^2 = T^*T$, for any $n \in \mathbb{N}_0$, we have

$$\langle \xi_n, |T|\xi_n\rangle = \langle \xi_n, T^*T\xi_n\rangle = \langle T\xi_n, T\xi_n\rangle = \|T\xi_n\|^2.$$

Thus,

$$\|T\|_2^2 = \sum_{n\geq 0} \|T\xi_n\|^2.$$

Finally, the equivalences (III.7) immediately follow from (III.6). The lemma is thus proved. $\square$

LEMMA III.11. *We have*

$$\|S+T\|_2 \leq \|S\|_2 + \|T\|_2 \qquad \forall S, T \in \mathcal{L}(\mathcal{H}). \tag{III.4.9}$$

PROOF. Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis. Using (III.6) and the Minkowski's inequality for series we get

$$\begin{aligned}\|S+T\|_2 = \left(\sum_{n\geq 0} \|(S+T)\xi_n\|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{n\geq 0} (\|S\xi_n\| + \|T\xi_n\|)^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n\geq 0} \|S\xi_n\|^2\right)^{\frac{1}{2}} + \left(\sum_{n\geq 0} \|T\xi_n\|^2\right)^{\frac{1}{2}} = \|S\|_2 + \|T\|_2,\end{aligned}$$

proving the lemma. $\square$

PROPOSITION III.11. *The following hold.*

(1) $\mathcal{L}^1$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$.
(2) $\|.\|_1$ *is a norm on* $\mathcal{L}^1$ *with respect to which* $\mathcal{L}^1$ *is a Banach space.*

PROOF. It follows from (III.2)–(III.3), (III.4) and (III.9) that $\mathcal{L}^2$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_2$ is a norm on $\mathcal{L}^2$. Moreover, by arguing as in the proof Proposition III.6 we can show that $\mathcal{L}^2$ is complete with respect to the norm $\|.\|_2$, i.e., $\mathcal{L}^2$ is a Banach space for this norm. □

REMARK III.10. Combining (III.2) and (III.4) shows that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| \leq \|T\|_2 \leq \|T\|_1.$$

Therefore, the inclusions $\mathcal{L}^1 \subset \mathcal{L}^2$ and $\mathcal{L}^2 \subset \mathcal{K}$ are continuous.

In addition, arguing as in the proof of Lemma III.12 yields the following.

LEMMA III.12. *The finite-rank operators are dense in* $\mathcal{L}^2$.

PROPOSITION III.12. *Let* S *and* T *be Hilbert-Schmidt operators. Then* ST *and* TS *are trace-class operators and we have*

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_2\|T\|_2,$$
$$\operatorname{Trace}(ST) = \operatorname{Trace}(TS).$$

PROOF. Let $ST = U|ST|$ be the polar decomposition of ST and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. By Proposition II.3 $|ST| = U^*ST$, and so using (III.6) we get

$$\|ST\|_1 = \sum_{n\geq 0}\langle \xi_n, U^*ST\xi_n\rangle = \sum_{n\geq 0}\langle SU\xi_n, T\xi_n\rangle \leq \sum_{n\geq 0}\|SU\xi_n\|\|T\xi_n\|.$$

Using Cauchy-Schwartz Inequality for sequences together with (III.6) we then get

$$\|ST\|_1 \leq \left(\sum_{n\geq 0}\|SU\xi_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 0}\|T\xi_n\|^2\right)^{\frac{1}{2}} = \|SU\|_2\|T\|_2 \leq \|S\|_2\|U\|\|T\|_2 \leq \|S\|_2\|T\|_2,$$

Notice that by Proposition II.3 and (III.3) we have $\|SU\|_2 \leq \|S\|_2\|U\| \leq \|S\|_2$. Thus,

$$\|ST\|_1 \leq \|S\|_2\|T\|_2. \tag{III.4.10}$$

This proves that ST is trace-class. Moreover, combining (III.10) with (III.12) yields

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_2\|T\|_2,$$

proving (III.12).

It follows from all this that, if we fix $S \in \mathcal{L}^2$, then both $T \to \operatorname{Trace}(ST)$ and $T \to \operatorname{Trace}(TS)$ are continuous linear forms on $\mathcal{L}^2$. Moreover, as finite-rank operators are trace-class, it follows from (III.14) that these linear forms agree on finite-rank operators. Since the latter are dense in $\mathcal{L}^2$ by Lemma III.12, it follows that $\operatorname{Trace}(ST) = \operatorname{Trace}(TS)$ for all $T \in \mathcal{L}^2$, completing the proof. □

For $S, T \in \mathcal{L}^2$ we define

$$(S, T) := \operatorname{Trace}(ST).$$

This defines a bilinear form on $\mathcal{L}^2$. If $S \in \mathcal{L}^2$, then by (III.4) its adjoint S^* is in $\mathcal{L}^2$ too. Therefore, for $S, T \in \mathcal{L}^2$ we may also define

$$\langle S, T\rangle_{\mathcal{L}^2} := \operatorname{Trace}(S^*T). \tag{III.4.11}$$

This defines a Hermitian form on $\mathcal{L}^2$. Moreover, as $T^*T = |T|^2$, using (III.6) we get

$$\langle T, T\rangle_{\mathcal{L}^2} = \operatorname{Trace} T^*T = \operatorname{Trace} |T|^2 = \|T\|_2^2.$$

Since $\|.\|_2$ is a norm on $\mathcal{L}^2$ this proves that $\langle ., .\rangle_{\mathcal{L}^2}$ is actually a positive-definite inner product on $\mathcal{L}^2$ whose associated norm is just the Hilbert-Schmidt norm. Combining this with Proposition III.11 we obtain:

PROPOSITION III.13. *$\mathcal{L}^2$ is a Hilbert space with respect to the inner product (III.11).*

For $S \in \mathcal{L}^2$ denote by $\langle S, .\rangle_{\mathcal{L}^2}$ the linear form $T \to \langle S, T\rangle_{L^2}$. Similarly, let us denote by $(S, .)$ the linear form $T \to (S, T)$. By definition of $\langle ., .\rangle_{\mathcal{L}^2}$ we have $(S, .) = \langle S^*, .\rangle$. Since $\langle ., .\rangle$ is a Hilbert-space inner product on $\mathcal{L}^2$. The map $S \to \langle S, .\rangle_{\mathcal{L}^2}$ is an isometric *antilinear* isomorphism from $\mathcal{L}^2$ onto $(\mathcal{L}^2)'$. Since (III.1) implies that $S \to S^*$ is an isometric *antilinear* isomorphism of $\mathcal{L}^2$ we obtain:

PROPOSITION III.14. *The map $\mathcal{L}^2 \ni S \to (S, .) \in (\mathcal{L}^2)'$ is an isometric linear isomorphism from $\mathcal{L}^2$ onto $(\mathcal{L}^2)'$.*

III.5. Integral Operators

Let (X, μ) be a σ-finite measured space such that $L^2_\mu(X)$ is separable. Let $K(x, y) \in L^2_{\mu\otimes\mu}(X \times X)$. For $f \in L^2_\mu(X)$ define

$$T_K f(x) = \int_X K(x, y) f(y) d\mu(y).$$

The function $T_K f(x)$ is measurable and by Cauchy-Schwartz's Inequality,

$$|T_K(x)|^2 \leq \int_X |K(x, y)|^2 d\mu(y) \int_X |f(x)|^2 d\mu(x) = \|f\|^2_{L^2} \int_X |K(x, y)|^2 d\mu(y),$$

and hence

$$\int_X |T_K(x)|^2 d\mu(x) \leq \|f\|^2_{L^2} \int_X \left(\int_X |K(x, y)|^2 d\mu(y) \right) d\mu(x) = \|f\|^2_{L^2} \|K\|^2_{L^2}$$

Therefore the map $T_K : f \to T_K f$ is a continuous endomorphism of $L^2_\mu(X)$. Such an operator is called a *integral operator.*

EXAMPLE III.11. Let φ and ψ be in $L^2_\mu(X)$. Then, for any $f \in L^2_\mu(X)$,

$$(\varphi \otimes \psi^*) f(x) = \langle \psi, f\rangle \varphi(x) = \varphi(x) \int_X \overline{\psi(y)} f(y) d\mu(y) = T_{\varphi\otimes\overline{\psi}} f(x),$$

where $\varphi \otimes \overline{\psi}$ is the element of $L^2_{\mu\otimes\mu}(X \times X)$ defined by

$$(\varphi \otimes \overline{\psi})(x, y) = \varphi(x)\overline{\psi(y)}.$$

This shows that any rank 1 operator is an integral operator, and hence by linearity any finite rank operator is an integral operator.

PROPOSITION III.15. *Let K and K' be in $L^2_{\mu\otimes\mu}(X \times X)$. Then*

$$T_K^* = T_{K^*} \qquad \text{and} \qquad T_K T_{K'} = T_{K*K'},$$

where K^* *and* $K * K'$ *are the functions in* $L^2_{\mu\otimes\mu}(X\times X)$ *defined by*

$$K^*(x,y)=\overline{K(y,x)}, \tag{III.5.1}$$

$$K*K'(x,y)=\int_X K(x,z)K'(z,y)d\mu(z). \tag{III.5.2}$$

Proof. It is immediate from its definition that $K^*(x,y)$ is in $L^2_{\mu\otimes\mu}(X\times X)$. Moreover, for any f and g in $L^2_\mu(X)$, we have

$$\begin{aligned}\langle f,T_Kg\rangle &= \int_X \overline{f(x)}T_Kg(x)d\mu(x)=\int_X\overline{f(x)}\left(\int_X K(x,y)g(y)d\mu(y)\right)d\mu(x)\\ &= \int_X\left(\int_X\overline{K^*(y,x)f(x)}d\mu(x)\right)g(y)d\mu(y)=\int_X\overline{T_{K^*}f(y)}g(y)d\mu(y)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =\langle T_{K^*}f,g\rangle,\end{aligned}$$

that is, T_{K^*} is the adjoint of T_K.

Next, as $K(x,y)$ and $K'(x,y)$ both are in $L^2_{\mu\otimes\mu}(X\times X)$ the function $K*K'(x,y)$ defined by (III.2) is a well defined measurable function. This is in fact an element of $L^2_{\mu\otimes\mu}(X\times X)$, for we have

$$|K*K'(x,y)|^2\leq\int_X|K(x,z)||K'(z,x)|d\mu(z)\leq\int_X|K(x,z)|^2d\mu(z)\int_X|K'(z,y)|^2d\mu(z),$$

and hence

$$\begin{aligned}&\int_{X\times X}|K*K'(x,y)|^2d\mu(x)d\mu(y)\leq\\ &\qquad\qquad\int_X\int_X|K(x,z)|^2d\mu(z)d\mu(x)\int_X\int_X|K'(z,y)|^2d\mu(z)d\mu(y)<\infty.\end{aligned}$$

Moreover, for any $f\in L^2_\mu(x)$, we have

$$\begin{aligned}T_KT_{K'}f(x)&=\int_X K(x,z)(T_{K'}f)(z)d\mu(z)=\int_X K(x,z)\left(\int_X K'(z,y)f(y)d\mu(y)\right)d\mu(z)\\ &=\int_X\left(\int_X K(x,z)K'(z,y)d\mu(z)\right)f(y)d\mu(y)=\int_X K*K'(x,y)f(y)d\mu(y)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =T_{K*K'}f(x),\end{aligned}$$

which shows that $T_KT_{K'}=T_{K*K'}$. The proof is complete. □

Proposition III.16. *The following hold.*

1. *For any* $K\in L^2_{\mu\otimes\mu}(X\times X)$ *the operator* T_K *is a Hilbert-Schmidt operator.*
2. *The map* $K\to T_K$ *is an isometric isomorphism from* $L^2_{\mu\otimes\mu}(X\times X)$ *onto* $\mathcal{L}^2(L^2_\mu(X))$.

Proof. Let $K\in L^2_{\mu\otimes\mu}(X\times X)$ and let $(\varphi_n)_{n\geq0}$ be an orthonormal basis of $L^2_\mu(X)$. By Lemma III.10 we have

$$\|T_K\|_2^2=\sum_{n\geq0}\|T_K\varphi_n\|^2=\sum_{n,m}|\langle\varphi_m,T_K\varphi_n\rangle|^2.$$

Notice that

$$\langle\varphi_m,T_K\varphi_n\rangle=\int_{X\times X}\overline{\varphi_m(x)}\varphi_n(x)K(x,y)d\mu(x)d\mu(y)=\langle\varphi_m\otimes\overline{\varphi_n},K\rangle_{L^2_{\mu\otimes\mu}(X\times X)}.$$

Since $(\varphi_m \otimes \overline{\varphi_n})_{m,n\geq 0}$ is an orthonormal basis of $L^2_\mu(X \times X)$, it follows that

$$\|T_K\|_2^2 = \sum_{n,m} |\langle \varphi_m \otimes \overline{\varphi_n}, K\rangle_{L^2_{\mu\otimes\mu}(X\times X)}|^2 = \|K\|^2_{L^2(X\times X)}.$$

Thus T_K is a Hilbert-Schmidt operator and $\|T_K\| = \|K\|_{L^2(X\times X)}$.

This shows that $K \to T_K$ is an isometric linear map from $L^2_{\mu\otimes\mu}(X \times X)$ to $\mathcal{L}^2(L^2_\mu(X))$. Lemma I.1 then insures us that this map is an isometric isomorphism from $L^2_{\mu\otimes\mu}(X\times X)$ onto its range which is a closed subspace of $\mathcal{L}^2(L^2_\mu(X))$. It follows from Example III.11 that any finite-rank operator on $L^2_\mu(X)$ is an integral operator. As by Lemma III.12 the finite-rank operators are dense in $\mathcal{L}^2$, we then deduce that the map $K \to T_K$ is onto. Therefore, it realizes an isometric isomorphism $L^2_{\mu\otimes\mu}(X \times X)$ onto $\mathcal{L}^2(L^2_\mu(X))$. The proof is complete. □

III.6. Trace Theorems for Integral Operators

In this section we prove a trace theorem for integral operators. Here we assume that X is a separable metrizable locally compact Hausdorff space (e.g., X is a manifold) and we also assume that μ is a Radon measure on X.

The assumption on the topology of X implies that X is σ-compact, and hence (X, μ) is a σ-finite measured space. This assumption also insures us that, for any compact $K \subset X$, the space $C_K(X)$ is separable. Together with the σ-compacity of X and the density of $C_c(X)$ in $L^2_\mu(X)$ this implies that $L^2_\mu(X)$ is separable.

Let us denote by $\operatorname{supp}\mu$ the support of μ. Recall that $X \setminus (\operatorname{supp}\mu)$ is the union set of all open sets $O \subset X$ such that $\mu(O) = 0$, and hence $\operatorname{supp}\mu$ is a closed subset of X.

Lemma III.13. *The following hold.*

(i) *The open set* $X \setminus (\operatorname{supp}\mu)$ *has measure* 0.
(ii) *The support of* $\mu_{|\operatorname{supp}\mu}$ *is equal to* $\operatorname{supp}\mu$.
(iii) *The support* $\mu \otimes \mu$ *is equal to* $(\operatorname{supp}\mu) \times (\operatorname{supp}\mu)$.

Proof. Set $V = X\setminus(\operatorname{supp}\mu)$. Since μ is a Radon measure and X is σ-compact, μ is a regular measure, and hence

$$\mu(V) = \sup\left\{\mu(K);\ K \subset V,\ K \text{ compact}\right\}. \tag{III.6.1}$$

Let K be a compact set contained in V, i.e., K is covered by the family of open subsets of measure 0. Since K is compact, there exist finitely many such open sets $O_1, \ldots, O_k$ that cover K, and hence K has measure zero. Thus any compact contained in V has measure zero. Combining this with (III.1) shows that $\mu(V) = 0$.

Set $E = \operatorname{supp}\mu$ and let O be an open subset of E such that $\mu(O) = 0$. Then there exists an open $O' \subset X$ such that $O = O' \cap E$. Then $O' \subset O \cup V$. As both O and V have measure zero, it follows that so does O'. Therefore O' is contained in V, and hence O must be the empty set. This shows that the only open subset of E that has measure zero is the empty set. Therefore the support of $\mu_{|E}$ is equal to E.

Set $W = X \times X \setminus \operatorname{supp}(\mu\otimes\mu)$. As $O \times X$ and $X \times O$ are open subsets of $X \times X$ on which $\mu \otimes \mu$ vanishes, we see that they both are contained in W. Conversely, if $(x, y) \in W$, then there exist an open neighborood O_1 of x in X and an open neighborhood O_2 of y in X such that $O_1 \times O_2 \subset W$. Then, using (i), we see that

$$\mu(O_1)\mu(O_2) = (\mu \otimes \mu)(O_1 \times O_2) \leq \mu(W) = 0,$$

Thus $\mu(O_1)$ or $\mu(O_1)$ must be zero, that is, O_1 or O_2 must be contained in V, and hence (x, y) is contained in $(V \times X) \cup (X \times V)$. It follows from all this that $W = (V \times X) \cup (X \times V)$, so taking complements shows that $\operatorname{supp}(\mu \otimes \mu)$ is equal to $(\operatorname{supp} \mu) \times (\operatorname{supp} \mu)$ agree. The proof is complete. $\square$

THEOREM III.2 (Duflo [**Du**]). *Let $K(x, y) \in L^2_{\mu\otimes\mu}(X \times X) \cap C(X \times X)$ and assume that the operator T_K is trace-class. Then the function $K(x, x)$ is in $L^1_\mu(X)$ and we have*

$$\operatorname{Trace} T_K = \int_X K(x, x) d\mu(x).$$

PROOF. Let $T_K = U|T_K|$ be the polar decomposition of T_K and let $(\xi_n)_{n\geq 0}$ be an orthonormal family in $L^2_\mu(X)$ such that $|T_K|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ set $\eta_n = U\xi_n$. As T_K is trace-class and the L^2-norms of the functions ξ_n are equal to 1, we have

$$\sum_{n\geq 0} \int_X \mu_n(T_K)|\xi_n(x)|^2 d\mu(x) = \sum_{n\geq 0} \mu_n(T_K) < \infty,$$

that is, the series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ converges normally in $L^1_\mu(X)$. Therefore, its sum is finite almost everywhere, i.e, the series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ converges almost evrywhere. Likewise, the series $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converges almost everywhere.

CLAIM. Let L be a compact subset of X. For any $\epsilon > 0$ there exists a compact $L' \subset L$ such that

(i) $\mu(L \setminus L') < \epsilon$.
(ii) For all $n \in \mathbb{N}_0$ the functions ξ_n and η_n are continuous on L'.
(iii) The series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converge uniformly on L'.

PROOF OF THE CLAIM. By Lusin's theorem (see, e.g., [**Fo**]), for any $n \in \mathbb{N}_0$, there exists a Borel set $E_n \subset L$ such that $\mu(L \setminus E_n) \leq 2^{-(n+1)}\epsilon$ and the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on E_n. Set $E = \bigcap_{n\geq 0} E_n$. Then E is a Borel set of L such that $\mu(L \setminus E) \leq \epsilon$ and *all* the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on E.

As the series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ converges almost everywhere on E, Egoroff's theorem (see, e.g., [**Fo**]) implies that there exists a Borel set $F \subset E$ such that $\mu(E \subset F) < \epsilon$ and the series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converge uniformly on F.

Since μ is a regular measure, there exists a compact subset L' of L such that $\mu(F \setminus L') \leq \epsilon$. As $L \setminus L' = (L \setminus E) \cup (E \setminus F) \cup (F \setminus \tilde{L})$, we then get

$$\mu(L \setminus L') \leq \mu(L \setminus E) + \mu(E \setminus F) + \mu(F \setminus L') \leq 3\epsilon.$$

In addition, as L' is contained in E and in F, all the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on L' and the series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converge uniformly on L'. The claim is thus proved. $\square$

Since X is σ-compact, there exists an increasing sequence $(L_j)_{j\geq 0}$ of compact sets such that $X = \bigcup_{j\geq 0} L_n$. For every $j \in \mathbb{N}_0$ the above claim insures us the existence of a compact $L'_j \subset L_j$ satisfies the conditions (i), (ii) and (iii) above with $L = L_j$, $L' = L'_j$ and $\epsilon = \frac{1}{j+1}$.

Let $j \in \mathbb{N}_0$. Set $\tilde{Y}_j = \bigcup_{k\leq j} L'_k$ and $Y_j = \operatorname{supp} \mu_{|Y_j}$. It follows from Lemma III.13 that $\mu(\tilde{Y}_j \setminus Y_j) = 0$ and $\operatorname{supp} \mu_{|Y_j} = Y_j$. Moreover, as

$$L_j \setminus Y_j = (L_j \setminus \tilde{Y}_j) \cup (\tilde{Y}_j \setminus Y_j) \subset (L_j \setminus L'_j) \cup (\tilde{Y}_j \setminus Y_j),$$

we see that

$$\mu(L_j \setminus Y_j) \leq \mu(L_j \setminus L'_j) + \mu(\tilde{Y}_j \setminus Y_j) \leq \frac{1}{j+1}. \tag{III.6.2}$$

In addition, as Y_j is contained in $\bigcup_{k\leq j} L'_k$ the following hold:

(a) All the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on Y_j.
(b) The series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converge uniformly on Y_j.

Set $Y = \bigcup_{j\geq 0} Y_j$. As $X \setminus Y = \bigcup_{j\geq 0}(X_j \setminus Y)$ and the sequence $(X_j \setminus Y)_{j\geq 0}$ is increasing, we have $\mu(X \setminus Y) = \lim_{j\to 0} \mu(X_j \setminus Y)$. As Y contains Y_j, using (III.2) we see that $\mu(X_j \setminus Y) \leq \mu(X_j \setminus Y_j) \leq \frac{1}{j+1}$. It then follows that $\mu(X \setminus Y) = 0$.

Claim. For all $(x, y) \in Y \times Y$, we have

$$K(x,y) = \sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}. \tag{III.6.3}$$

Proof. Let $j \in \mathbb{N}_0$. For all $p, q \in \mathbb{N}_0$ we have

$$\sum_{p\leq n\leq q} \mu_n(T)|\eta_n(x)\overline{\xi_n(y)}| \leq \Bigg(\sum_{p\leq n\leq q} \mu_n(T)|\xi_n(x)|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{p\leq n\leq q} \mu_n(T)|\eta_n(x)|^2\Bigg)^{\frac{1}{2}}.$$

Therefore, using the property (b) above we see that the series $\sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}$ converges uniformaly on $Y_j \times Y_j$.

For all $(x, y) \in Y_j \times Y_j$ set

$$K'(x,y) = \sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}.$$

Since by the property (a) all the functions $\eta_n(x)\overline{\xi_n(y)}$ are continuous on $Y_j \times Y_j$ and the above series converges uniformly on $Y_j \times Y_j$, this defines a continuous function on $Y_j \times Y_j$. Moreover, as $Y_j \times Y_j$ is compact, and hence has finite measure, since $\mu \otimes \mu$ is a Radon measure, the uniform convergence implies the convergence in L^2-norm.

On the other hand, as T_K is trace-class, the proof of Lemma III.8 shows that

$$T = \sum_{n\geq 0} (U\xi_n) \otimes \xi_n^* = \sum_{n\geq 0} \eta_n \otimes \xi_n^*, \tag{III.6.4}$$

Since by Remark III.10 the inclusion of $\mathcal{L}^2$ into $\mathcal{L}^1$ is continuous, it follows that the above series converges in $\mathcal{L}^2$-norm too. Combining this with Proposition III.16 we deduce that the series of the corresponding kernel functions converge to $K(x, y)$ in L^2-norm. As shown in Example III.11, for every $n \in \mathbb{N}_0$, the kernel function of $\eta_n \otimes \xi_n^*$ is equal to $\eta_n(x)\overline{\xi_n(y)}$. Therefore, the series $\sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}$ converges to $K(x, y)$ in L^2-norm. Since we already know that it converges to $K'(x, y)$ in L^2-norm on $Y_j \times Y_j$, it follows that

$$K(x,y) = K'(x,y) \qquad \text{almost everywhere on } Y_j \times Y_j.$$

Observe that both $K(x,y)$ and $K'(x,y)$ are continuous functions on $Y_j \times Y_j$. Therefore $W := \{(x,y) \in Y_j \times Y_j; K(x,y) \neq K'(x,y)\}$ is an open subset $Y_j \times Y_j$ of measure zero, and hence it is contained $V = X \setminus (\operatorname{supp} \mu \otimes \mu_{|Y_j \times Y_j})$. By construction $\operatorname{supp} \mu_{|Y_j} = Y_j$, so using Lemma III.13 we see that the support of $(\mu \otimes \mu)_{Y_j \times Y_j} = (\mu_{|Y_j}) \otimes (\mu_{|Y_j})$ is equal to $Y_j \times Y_j$. Therefore, V must be the empty set, and hence $K(x,y) = K'(x,y)$ for all $(x,y) \in Y_j \times Y_j$. Thus,

$$K(x,y) = \sum_{n \geq 0} \mu_n(T) \eta_n(x) \overline{\xi_n(y)} \qquad \forall (x,y) \in Y_j \times Y_j,$$

where the series converges uniformly on $Y_j \times Y_j$.

All this shows that the equality (III.3) holds on all the products $Y_j \times Y_j$. Since $Y \times Y = \bigcup_{j \geq 0} Y_j \times Y_j$ the claim follows. □

Since $\mu(X \setminus Y) = 0$, it follows from (III.3) that, almost everywhere on X,

$$K(x,x) = \sum_{n \geq 0} \mu_n(T) \eta_n(x) \overline{\xi_n(x)}. \tag{III.6.5}$$

Moreover, as

$$\int_X |\eta_n(x) \overline{\xi_n(x)}| d\mu(x) \leq \left(\int_X |\eta_n(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_X |\xi_n(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \leq 1,$$

we see that

$$\sum_{n \geq 0} \int_X \mu_n(T) |\eta_n(x) \overline{\xi_n(x)}| d\mu(x) \leq \sum_{n \geq 0} \mu_n(T) < \infty.$$

Therefore, the series in (III.5) converges in L^1-norm. This implies that $K(x,x)$ is contained in $L^1_\mu(X)$ and we have

$$\int_X K(x,x) d\mu(x) = \sum_{n \geq 0} \mu_n(T) \int_X \eta_n(x) \overline{\xi_n(x)} d\mu(x) = \sum_{n \geq 0} \mu_n(T) \langle \xi_n, \eta_n \rangle. \tag{III.6.6}$$

On the other hand, as explained in Example III.6, for every n, the trace of the projection $\eta_n \otimes \xi_n^*$ is equal to $\langle \xi_n, \eta_n \rangle$. Since in (III.4) the series converges in $\mathcal{L}^1$-norm and the functional $T \rightarrow \operatorname{Trace}(T)$ is continuous with respect to that norm, we deduce that

$$\operatorname{Trace} T_K = \sum_{n \geq 0} \mu_n(T) \operatorname{Trace}(\eta_n \otimes \xi_n^*) = \sum_{n \geq 0} \mu_n(T) \langle \xi_n, \eta_n \rangle.$$

Combining this with (III.6) proves that

$$\operatorname{Trace} T_K = \int_X K(x,x) d\mu(x).$$

The proof is complete. □

Remark III.12. If X is compact then the sole continuity of $K(x,y)$ insures us that $K(x,y)$ is square-integrable on $X \times X$ and $K(x,x)$ is integrable on X. However, in general this is not enough to insure us that T_K is trace-class (see, e.g., [**GK**, §10.3]). Thus in Theorem III.2 we cannot remove the assumption on T_K being trace-class (unless T_K is positive; see below).

When T_K we don't need to assume T_K to be trace-class, because we can make use of Mercer's theorem to prove:

Theorem III.3 ([**Du**]). *Let $K(x,y)\in L^2_{\mu\otimes\mu}(X\times X)\cap C(X\times X)$ be such that T_K is positive. Then*

(1) *$K(x,x)\geq 0$ for all $x\in X$.*
(2) *We have*

$$\operatorname{Trace} T_K = \int_X K(x,x)d\mu(x).$$

Thus,

$$T_K\in\mathcal{L}^1 \iff K(x,x)\in L^1_\mu(X).$$

Remark III.13. We refer to [**Br**] for generalizations of Duflo's theorems where the assumptions on the continuity of $K(x,y)$ are relaxed.

III.7. Banach Ideals

In the remainder of the chapter we shall present a detailed account on the theory of Calkin and Gohberg-Krein of operator ideals and operator ideals associated to symmetric norms. As we shall see these ideals play an important role in noncommutative geometry.

Most of the material that follows is taken from [**GK**] and [**Si**] (see also [**Co**, Chap. 4, Appendix C], [**GVF**, Section 7.C]).

This section is devoted to presenting the primary definitions and properties of Banach ideals. We start with basic facts about two-sided ideals in $\mathcal{L}(\mathcal{H})$.

Proposition III.17. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *For any $T\in\mathcal{L}(\mathcal{H})$,*

$$T\in\mathcal{I}\Longleftrightarrow |T|\in\mathcal{I}\Longleftrightarrow T^*\in\mathcal{I}.$$

(2) *Any $T\in\mathcal{I}$ can be written as*

$$T=T_1-T_2+i(T_3-T_4)\qquad \text{with } T_j\in\mathcal{I}\cap\mathcal{L}(\mathcal{H})_+.$$

Proof. Once (1) is proved the proof of (2) follows along the same lines as that of the proof of Lemma III.8. Thus, we only have to prove (1).

Let $T\in\mathcal{L}(\mathcal{H})$ have polar decomposition $T=U|T|$. If $|T|$ is in $\mathcal{I}$ then, as $\mathcal{I}$ is an ideal, $T=U|T|$ is in $\mathcal{I}$ too. Since by Proposition II.3 $|T|=U^*T$ we also see that if $|T|$ is in $\mathcal{I}$, then so is T.

It also follows from Proposition II.3 that $T^*=U^*TU$, and $T=(U^*TU^*)^*=UT^*U$. Therefore T is in $\mathcal{I}$ if and only if is T^* in $\mathcal{I}$. The proof is complete. □

Proposition III.18. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *If $\mathcal{I}\supsetneq\{0\}$, then every finite-rank operator is contained in $\mathcal{I}$.*
(2) *If $\mathcal{I}\subsetneq\mathcal{L}(\mathcal{H})$, then every operator in $\mathcal{I}$ is compact.*

Proof. Assume $\mathcal{I}\supsetneq\{0\}$. Since the finite-rank operators are linear combinations of rank 1 operators $\xi\otimes\eta^*$, $\xi,\eta\neq 0$, in order to prove (1) it is enough to show that any such projection is contained in $\mathcal{I}$.

Let $\xi,\eta\in\mathcal{H}\setminus\{0\}$ and let $T\in\mathcal{I}\setminus\{0\}$. Since $T\neq 0$ there exists $\xi'\in\mathcal{H}\setminus\{0\}$ such that $\eta':=T\xi'\neq 0$. Set $A=\xi\otimes\xi^{'*}$ and $B=\eta'\otimes\eta^*$. Then the operator ATB is contained in $\mathcal{I}$ and is equal to $(\xi\otimes\eta^{'*})T(\xi\otimes\eta^*)=\langle\eta',T\xi'\rangle(\xi\otimes\eta^*)=\|\eta'\|^2(\xi\otimes\eta^*)$. Since $\eta'\neq 0$ it follows that $\xi\otimes\eta^*$ is contained in $\mathcal{I}$, proving (1).

Suppose now that $\mathcal{I}$ contains a non-compact operator T. By Proposition II.7 and Proposition III.17 the operator $|T|$ too is non-compact and contained in $\mathcal{I}$.

Therefore, possibly by replacing T by $|T|$, we may assume T positive. For $\lambda > 0$ set $\Pi_\lambda = 1_{[\lambda,\infty)}(T)$. If $g(t) := t^{-1}1_{[\lambda,\infty)}$, then $g(T)$ is a bounded operator. As $\Pi_\lambda = Tg(T)$ it follows that Π_λ is contained in $\mathcal{I}$.

As $\|T - T\Pi_\lambda(T)\| = \|1_{[0,\lambda)}(T)\| \leq \lambda$, we see that $T\Pi_\lambda(T)$ converges to T in norm as $\lambda \to 0^+$. Since T is non-compact, it follows there is at least one $\lambda > 0$ such that $T\Pi_\lambda$ does not have finite rank. Then Π_λ does not have finite rank.

Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$, let $(\eta_n)_{n\geq 0}$ be an orthonormal basis of $\operatorname{im}\Pi_\lambda$, and let $V \in \mathcal{L}(\mathcal{H})$ be such that $V\xi_n = \eta_n$. As $V^*\eta_n = \xi_n$ for all $n \in \mathbb{N}$, we see that $V^*\Pi_\lambda V = 1$. Thus $1 \in \mathcal{I}$, which implies that $\mathcal{I} = \mathcal{L}(\mathcal{H})$. Therefore, if $\mathcal{I} \subsetneq \mathcal{L}(\mathcal{H})$, then $\mathcal{I}$ cannot contain any non-compact operator, i.e., T is contained in $\mathcal{K}$. The proof is complete. □

The previous proposition shows that, among non-trivial ideals of $\mathcal{L}(\mathcal{H})$, the ideal of finite-rank operators is minimal and the ideal of compact operators is maximal. Since the former is the closure of the latter in $\mathcal{L}(\mathcal{H})$ we obtain:

Corollary III.1. *The only closed non-trivial two-sided ideal of $\mathcal{L}(\mathcal{H})$ is $\mathcal{K}$.*

Definition III.14. *A Banach ideal is a two-ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ which is equipped with a norm $\|.\|_{\mathcal{I}}$ such that*

(i) *$\mathcal{I}$ is a Banach space for $\mathcal{I}$.*
(ii) *We have*

$$\|ATB\|_{\mathcal{I}} \leq \|A\|\|T\|_{\mathcal{I}}\|B\| \qquad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{III.7.1}$$

Example III.15. $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ are Banach ideals for the operator norm $\|.\|$.

Example III.16. It follows from (III.5) and Proposition III.6 that $\mathcal{L}^1$ is a Banach ideal for the norm $\|.\|_1$. Likewise, using (III.3) and Proposition III.11, we see that $\mathcal{L}^2$ is a Banach ideal for the Hilbert-Schmidt norm $\|.\|_2$.

In the sequel we let $\mathcal{I}$ be a Banach ideal with norm $\|.\|_{\mathcal{I}}$. We assume $\mathcal{I}$ non-trivial, so by Proposition III.18 all the finite-rank operators are contained in $\mathcal{I}$ and all the elements of $\mathcal{I}$ are compact operators.

Lemma III.14. *Let $T \in \mathcal{I}$ and let $S \in \mathcal{K}$.*

(i) *If $\mu_n(S) \leq \mu_n(T)$ $\forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.*
(ii) *If $\mu_n(S) = \mu_n(T)$ $\forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}$.*

Proof. We only have to prove (i), since it implies (ii). Thus, let us assume that $\mu_n(S) \leq \mu_n(T)$ $\forall n \in \mathbb{N}_0$, and let $T = U|T|$ and $S = V|S|$ be the respective polar decompositions of T and S. Let $(\xi_n)_{n\geq 0}$ and $(\eta_n)_{n\geq 0}$ be orthonormal families in $\mathcal{H}$ such that $|T|\xi_n = \mu_n(T)\xi_n$ and $|S|\eta_n = \mu_n(S)\eta_n$ for all $n \in \mathbb{N}_0$. Let $C \in \mathcal{L}(\mathcal{H})$ be such that $C = 0$ on $\ker|S|$ and $C\eta_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\xi_n$ for all $n \in \mathbb{N}_0$ such that $\mu_n(T) > 0$ (i.e., ξ_n is in $\operatorname{im}|S| = (\ker|S|)^\perp$. This defines bounded operator of norm ≤ 1,since by assumption $\mu_n(S) \leq \mu_n(T)$ for all $n \in \mathbb{N}_0$. As $C^*\xi_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\eta_n$ for all $n \in \mathbb{N}_0$, we see that $C^*|T|C = |S|$.

By Proposition II.3 we know that $|T| = U^*T$, so we have

$$VC^*U^*TC = VC^*|T|C = V|S| = S.$$

Therefore S is contained in $\mathcal{I}$ and, by (III.1), we have $\|S\|_{\mathcal{I}} \leq \|V\|\|C^*\|\|U\|\|T\|_{\mathcal{I}}\|C\|$. Since the operator norms of U, V and C are ≤ 1, it follows that $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$, as claimed. □

Combining this lemma with (III.4) and (III.6) we see that, for any $T \in \mathcal{I}$,

$$\|T\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}},$$
$$\|U^*TU\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}.$$

Proposition III.19. *There exists a constant $c > 0$ such that*

$$\|T\|_{\mathcal{I}} = c\|T\| \qquad \forall T \in \mathcal{R}_1. \tag{III.7.2}$$

Furthermore, we have

$$c\|T\| \leq \|T\|_{\mathcal{I}} \qquad \forall T \in \mathcal{I}. \tag{III.7.3}$$

Proof. Let $R \in \mathcal{R}_1$ be such that $\|R\| = 1$ and set $c = \|R\|_{\mathcal{I}}$. It follows from (III.2) that $\mu_0(R) = \|R\| = 1$ and $\mu_n(R) = 0$ for $n \geq 1$. Likewise, if $S \in \mathcal{R}_1$, then $\mu_0(S) = \|S\|$ and $\mu_n(S) = 0$ for $n \geq 1$, so the operators S and $\|S\|R$ have the same characteristic values. Lemma III.14 then implies that $\|S\|_{\mathcal{I}} = \|S\|\|R\|_{\mathcal{I}} = c\|S\|$.

Let $T \in \mathcal{I}$. Then $\mu_0(T) = \|T\| = \mu_0(\|T\|R)$ and $\mu_n(T) \geq 0 = \mu_n(\|T\|R)$ for $n \geq 1$, so by Lemma III.14 we have $\|T\|_{\mathcal{I}} \geq \|(\|T\|R)\|_{\mathcal{I}} = c\|T\|$, as claimed. □

Because the norm $\|.\|_{\mathcal{I}}$ on rank-one operators is constant, we sometimes require the normalization,

$$\|T\|_{\mathcal{I}} = \|T\| \quad \text{for any operator } T \text{ of rank } 1. \tag{III.7.4}$$

In this case, the inequality (III.3) holds with $c = 1$.

Proposition III.20. *Any other Banach norm on $\mathcal{I}$ satisfying (III.1) is equivalent to $\|.\|_{\mathcal{I}}$.*

Proof. Let $\|.\|'_{\mathcal{I}}$ be another Banach norm on $\mathcal{I}$ satisfying (III.1) and let $|.|_{\mathcal{I}}$ be the norm on $\mathcal{I}$ defined by

$$|T|_{\mathcal{I}} := \sup\{\|T\|_{\mathcal{I}}, \|T\|'_{\mathcal{I}}\} \qquad \forall T \in \mathcal{I}.$$

Let $(T_n)_{n\geq 0}$ be a Cauchy sequence in $(\mathcal{I}, |.|_{\mathcal{I}})$, i.e., it is a Cauchy sequence both in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. It thus converges in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and in $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. The limits may be different. However, using (III.3) we see that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $(\mathcal{L}(\mathcal{H}), \|.\|)$ and its limit in $(\mathcal{L}(\mathcal{H}), \|.\|)$ agrees with the limits in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. Thus, the last two limits are equal and $(T_n)_{n\geq 0}$ converges in $(\mathcal{I}, |.|_{\mathcal{I}})$. This shows that $(\mathcal{I}, |.|_{\mathcal{I}})$ is a Banach space.

Notice that the identity map is continuous from $(\mathcal{I}, |.|_{\mathcal{I}})$ to $(\mathcal{I}, \|.\|_{\mathcal{I}})$. Since this is a bijection and both $(\mathcal{I}, |.|_{\mathcal{I}})$ to $(\mathcal{I}, \|.\|_{\mathcal{I}})$ are Banach spaces, the open mapping theorem insures us that its inverse is continuous. Therefore $|.|_{\mathcal{I}}$ and $\|.\|_{\mathcal{I}}$ are equivalent norms. Likewise, the norms $|.|_{\mathcal{I}}$ and $\|.\|'_{\mathcal{I}}$ are equivalent, so $\|.\|_{\mathcal{I}}$ and $\|.\|'_{\mathcal{I}}$ are equivalent norms, proving the proposition. □

As we shall now see the separability of the topology of $\mathcal{I}$ defined by the norm $\|.\|_{\mathcal{I}}$ is intimately related to the density of finite-rank operators.

Definition III.17. *$\mathcal{I}^0$ is the closure in $\mathcal{I}$ of the the ideal $\mathcal{R}_\infty$ of finite-rank operators.*

Since $\mathcal{R}_\infty$ is a two-sided ideal, $\mathcal{I}^0$ can easily be seen to be a Banach ideal for the norm of $\mathcal{I}$.

Let $T \in \mathcal{K}$ have polar decomposition $T = U|T|$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Then, by Proposition III.5,

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*, \tag{III.7.5}$$

where the series converges in $\mathcal{K}$. Any series of the form (III.5) is called a *Schmidt series* for T.

LEMMA III.15. *Let $T \in \mathcal{K}$. Then the following are equivalent:*

(i) *T is contained in $\mathcal{I}^0$.*
(ii) *Any Schmidt series for T converges in $\mathcal{I}$ to T.*
(iii) *There is a Schmidt series for T which converges in $\mathcal{I}$.*

PROOF. It is clear that (ii) implies (iii). Moreover, if there is a Schmidt series for T converging in $\mathcal{I}$ then, as it converges to T in $\mathcal{K}$, using (III.3) we see that its sum is equal to T. Thus T is contained in $\mathcal{I}$ and is the sum of a series of finite-rank operators, hence T is an element of $\mathcal{I}^0$.

Suppose now that T is in $\mathcal{I}$. For any $N \in \mathbb{N}$ set

$$T_N := T - \sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^* = \sum_{n\geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^*.$$

As $\sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ has rank $\geq N$, it is immediate that

$$\|T_N\|_{\mathcal{I}} \geq \inf\Big\{ \|T - R\|_{\mathcal{I}};\ R \in \mathcal{R}_N \Big\}. \tag{III.7.6}$$

It follows from Proposition II.3 the operator U^*U is the orthogonal projection onto $(\ker T)^\perp = (\ker |T|)^\perp = \operatorname{im} |T|$, so $U^*U\xi_n = \xi_n$ if $\mu_n(T) \neq 0$. Thus,

$$(T_N)^*T_N = \sum_{n\geq N} \mu_n(T)^2 \xi_n \otimes \xi_n^* \qquad \text{and} \qquad |T_N| = \sum_{n\geq N} \mu_n(T)\xi_n \otimes \xi_n^*.$$

Using the min-max principle we then see that

$$\mu_n(T_N) = \mu_{n+N}(T) \qquad \forall n \in \mathbb{N}. \tag{III.7.7}$$

Let $R \in \mathcal{R}_N$. Then (III.2) implies that $\mu_N(R) = 0$, so using (III.7) we get

$$\mu_n(T_N) = \mu_{n+N}(T) \leq \mu_n(T - R) + \mu_N(R) = \mu_n(T - R).$$

Therefore, applying Lemma III.14 we see that $\|T_N\|_{\mathcal{I}} \leq \|T - R\|_{\mathcal{I}}$ for all $R \in \mathcal{R}_N$. Combining this with (III.6) then shows that

$$\Big\| \sum_{n\geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^* \Big\|_{\mathcal{I}} = \inf\Big\{ \|T - R\|_{\mathcal{I}}; R \in \mathcal{R}_N \Big\}.$$

This implies that T is a limit of finite-rank operators in $\mathcal{I}$ (i.e., T is in $\mathcal{I}^0$) if and only if the Schmidt series $\sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges to T in $\mathcal{I}$. The proof is complete. □

LEMMA III.16. *The Banach ideal $\mathcal{I}^0$ is separable.*

PROOF. Without any loss of generality we may assume that in (III.2)–(III.3) the constant c is equal to 1. Let $(\zeta_k)_{k\geq 0}$ be a countable dense subset of $\mathcal{H}$. Let

$\xi, \eta \in \mathcal{H}$. For any $\epsilon > 0$ there exist $k, l \in \mathbb{N}_0$ such that $\|\xi - \zeta_k\| \leq \epsilon$ and $\|\eta - \zeta_l\| \leq \epsilon$. Then using (III.2) we get

$$\begin{aligned}
\text{(III.7.8)} \quad \|\xi \otimes \eta^* - \zeta_k \otimes \zeta_l^*\|_{\mathcal{I}} &\leq \|(\xi - \zeta_k) \otimes \eta^*\|_{\mathcal{I}} + \|\zeta_k \otimes (\eta - \zeta_l)^*\|_{\mathcal{I}} \\
&\leq \|(\xi - \zeta_k) \otimes \eta^*\| + \|\zeta_k \otimes (\eta - \zeta_l)^*\| \\
&\leq \|\xi - \zeta_k\|\|\eta\| + \|\zeta_k\|\|\eta - \zeta_l\| \leq \epsilon\|\eta\| + (\epsilon + \|\xi\|)\epsilon.
\end{aligned}$$

Let $\mathcal{D}$ be the set of operators of the form,

$$\sum_{(k,l) \in K \times L} \zeta_k \otimes \zeta_l^*,$$

where K and L range over all finite subsets of $\mathbb{N}_0$. Then $\mathcal{D}$ is a countable subset of $\mathcal{R}_\infty$. As any operator in $\mathcal{R}_\infty$ is a finite sum of rank one operators $\xi \otimes \eta^*$, it follows from (III.8) that, for any $T \in \mathcal{R}_\infty$ and for any $\epsilon > 0$, there exists $R \in \mathcal{D}$ such that $\|T - R\|_{\mathcal{I}} < \epsilon$. Combining this with the density of $\mathcal{R}_\infty$ in $\mathcal{I}^0$ we deduce that $\mathcal{D}$ is dense in $\mathcal{I}^0$. Since $\mathcal{D}$ is countable, this proves that $\mathcal{I}^0$ is separable. □

Proposition III.21. *The following are equivalent:*

(1) *The finite-rank operators are dense in $\mathcal{I}$, i.e., $\mathcal{I} = \mathcal{I}_0$.*
(2) *$\mathcal{I}$ is separable.*

Proof. It immediately follows from Lemma III.16 that if $\mathcal{I} = \mathcal{I}^0$ then $\mathcal{I}$ is separable.

Conversely, suppose that $\mathcal{I}_0 \subsetneq \mathcal{I}$. Let $T \in \mathcal{I} \setminus \mathcal{I}_0$. Since $\mathcal{I}$ and $\mathcal{I}_0$ both are ideals, using Proposition III.17 we see that $|T|$ is in $\mathcal{I}$, but is not in $\mathcal{I}$. Therefore, possibly by replacing T by $|T|$ we may assume T positive.

Since T is in $\mathcal{I} \setminus \mathcal{I}_0$ and is positive, Lemma III.15 implies that there is a Schmidt series $\sum_{n \geq 0} \mu_n(T) \xi_n \otimes \xi_n^*$ which does not converges in $\mathcal{I}$. As $\mathcal{I}$ is a Banach space, this implies that the series does not satisfy Cauchy's criterion, so there exists $\delta > 0$ and an increasing sequence $(n_k)_{k \geq 0} \subset \mathbb{N}_0$ such that

$$\text{(III.7.9)} \qquad \Big\| \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \Big\|_{\mathcal{I}} \geq \delta \qquad \forall k \in \mathbb{N}_0.$$

For any sequence $a = (a_k)_{k \geq 0} \in \{0, 1\}^{\mathbb{N}_0}$ we set

$$T_a := \sum_{k=0}^{\infty} a_k \Big(\sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \Big) = \sum_{a_k \neq 0}^{\infty} \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

If we let Π_a be the orthogonal projection onto the closure of the vector space spanned by $\bigcup_{a_k \neq 0} \{\xi_n;\ n_k \leq n < n_{k+1}\}$, then $T_a = \Pi_a T$. Therefore, the operator T_a is in $\mathcal{I}$.

Let $b = (b_k)_{k \geq 0} \in \{0, 1\}^{\mathbb{N}_0}$ be such that $b \neq a$, i.e., there exists $k \in \mathbb{N}_0$ such that $b_k \neq a_k$. Set $\Pi_k = \sum_{n=n_k}^{n_{k+1}-1} \xi_n \otimes \xi_n^*$. Then

$$\Pi_k (T_b - T_a) = (b_k - a_k) \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* = \pm \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

Combining this with (III.9) we get

$$\delta \leq \|\Pi_k (T_b - T_a)\|_{\mathcal{I}} \leq \|\Pi_k\| \|T_b - T_a\|_{\mathcal{I}} = \|T_b - T_a\|_{\mathcal{I}}.$$

Since $\{0,1\}^{\mathbb{N}}$ is not countable, it follows that no countable subset of $\mathcal{I}$ can be dense, so $\mathcal{I}$ is not separable if $\mathcal{I}_0 \subsetneq \mathcal{I}$. Equivalently, if $\mathcal{I}$ is separable, then $\mathcal{I} = \mathcal{I}_0$. The proof is complete. $\square$

The following result hows that, among the non-trivial Banach ideals, the ideal $\mathcal{L}^1$ of trace-class operators is minimal.

Proposition III.22. *There is a continuous inclusion,*

$$\mathcal{L}^1 \subset \mathcal{I}^0.$$

In fact, if the normalization (III.4) holds, then

$$\|T\|_{\mathcal{I}} \leq \|T\|_1 \qquad \forall T \in \mathcal{I}. \tag{III.7.10}$$

Proof. We may assume that the normalization (III.4) holds, so that we can take $c = 1$ in (III.3). Let $T \in \mathcal{L}^1$ and let $\sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ be a Schmidt series for T as in (III.5). Using (III.1) and (III.3)) we see that, for all N and p in $\mathbb{N}$, we have

$$\begin{aligned} \Big\| \sum_{N\leq n\leq N+p} \mu_n(T)(U\xi_n) \otimes \xi_n^* \Big\|_{\mathcal{I}} &\leq \sum_{N\leq n\leq N+p} \mu_n(T)\|U\|\|\xi_n \otimes \xi_n^*\|_{\mathcal{I}} \\ &\leq \sum_{N\leq n\leq N+p} \mu_n(T)\|\xi_n \otimes \xi_n^*\| \leq \sum_{N\leq n\leq N+p} \mu_n(T). \end{aligned} \tag{III.7.11}$$

Since $\sum_{n\geq 0} \mu_n(T) < \infty$ it follows that the series $\sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges in $\mathcal{I}$. Lemma III.15 then insures us that T is contained in $\mathcal{L}^0$ and the Schmidt series converges to T in $\mathcal{I}$. Therefore, using (III.11), we get

$$\|T\|_{\mathcal{I}} = \Big\| \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^* \Big\|_{\mathcal{I}} \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1.$$

This proves (III.10) when T is in $\mathcal{L}^1$ and shows there is a continuous inclusion of $\mathcal{L}^1$ in $\mathcal{I}^0$. In addition, if $T \in \mathcal{I} \setminus \mathcal{L}^1$, then $\|T\|_1 = \infty$ and (III.10) holds trivially, so (III.10) holds for all $T \in \mathcal{I}$. The proof is complete. $\square$

III.8. Symmetric norms

In the sequel we denote by l_f the vector space of sequences $a = (a_n)_{n\geq 0}$ of complex numbers that have finite support (i.e., $a_n = 0$ for n large enough). We denote by l_0 the space of sequences $(a_n)_{n\geq 0}$ of complex numbers such that

$$\lim_{n\to\infty} a_n = 0.$$

For any sequence $a = (a_n)_{n\geq 0}$ in l_0 we denote by $\sigma(a) = (\sigma_N(a))_{N\geq 1})$ the sequence defined by

$$\sigma_N(a) := \sum_{n<N} a_n \qquad \forall N \in \mathbb{N}.$$

In addition, for any $a \in l_0$ we denote by $a^* = (a_n^*)_{n\geq 0}$ the sequence defined by

$$a_n^* = \inf_{\substack{J\subset\mathbb{N}_0 \\ |J|=n}} \sup_{j\in J} |a_j| \qquad \forall n \in \mathbb{N}_0.$$

In other word, the sequence $(a_n^*)_{n\geq 0}$ is the sequence obtained by re-ordering the sequence $(|a_n|)_{n\geq 0}$ into a non-increasing sequence. In particular, for any $N \in \mathbb{N}$, we always have

$$|\sigma_N(a_n)| \leq \sum_{n<N} |a_n| \leq \sigma_N(a^*).$$

It can also be shown (see [**Si**, Lem. 1.8]) that, for all $a, b \in l_f$,

$$|\sum_{n<N} a_n b_n| \leq \sum_{n<N} a_n^* b_n^* \qquad \forall N \in \mathbb{N}. \tag{III.8.1}$$

DEFINITION III.18. *Let Φ be a norm on l_f. We say that Φ is symmetric when*

$$\Phi(a) = \Phi(a^*) \qquad \forall a \in l_f.$$

REMARK III.19. It is not difficult to check that a norm Φ on l_f is symmetric if and only if it satisfies the following two conditions:

(i) For any sequence $(a_n)_{n\geq 0}$ in l_f and any bijection $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$, we have

$$\Phi\left((a_{\sigma(n)})_{n\geq 0}\right) = \Phi\left((a_n)_{n\geq 0}\right).$$

(ii) For any sequence $(a_n)_{n\geq 0}$ in l_f and any sequence $(\theta_n)_{n\geq 0} \subset [0, 2\pi)$, we have

$$\Phi\left((e^{i\theta_n} a_n)_{n\geq 0}\right) = \Phi\left((a_n)_{n\geq 0}\right).$$

EXAMPLE III.20. For $p \in [1, \infty)$ the p-norm Φ_p on l_f is defined by

$$\Phi_p(a) = \left(\sum_{n\geq 0} |a_n|^p\right)^{\frac{1}{p}} \qquad \forall a = (a_n)_{n\geq 0} \in l_f.$$

For $p = \infty$ we define the Φ_∞-norm by

$$\Phi_\infty(a) = \sup_{n\geq 0} |a_n| \qquad \forall a = (a_n)_{n\geq 0} \in l_f. \tag{III.8.2}$$

All the p-norms are symmetric norms on l_f.

Let Φ be a symmetric norm on l_f.

LEMMA III.17 (Markus; see [**GK**, Lem. 3.1], [**Si**, Thm. 1.9]). *Let $a, b \in l_f$. Then*

$$\Big(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N}\Big) \Longrightarrow \Phi(a) \leq \Phi(b). \tag{III.8.3}$$

If follows from Markus' lemma that if $a_n^* \leq b_n^*$ for all $n \in \mathbb{N}_0$, then $\Phi(a) \leq \Phi(b)$. In particular, if $a = (a_n)_{\geq 0}$ is a sequence in l_0, then

$$\Phi(a_0, \ldots, a_{N-1}, 0, 0, \ldots) \leq \Phi(a_0, \ldots, a_N, 0, 0, \ldots) \qquad \forall N \in \mathbb{N}.$$

This means that $(\Phi(a_0, \ldots, a_N, 0, 0, \ldots))_{N\geq 0}$ is a non-decreasing sequence of non-negative numbers, so it admits a limit as $N \to \infty$. We then set

$$\Phi(a) = \lim_{N\to\infty} \Phi(a_0, \ldots, a_N, 0, 0, \ldots) = \sup_{N\geq 1} \Phi(a_0, \ldots, a_N, 0, 0, \ldots) \tag{III.8.4}$$

This extends Φ to a function $\Phi : l_0 \to [0, \infty]$.

It is not hard to check that

$$\Phi(a) = 0 \Longrightarrow a = 0, \tag{III.8.5}$$

$$\Phi(\lambda a) = |\lambda|\Phi(a) \qquad \forall a \in l_0 \ \forall \lambda \in \mathbb{C}, \tag{III.8.6}$$

$$\Phi(a+b) \leq \Phi(a) + \Phi(b) \qquad \forall a, b \in l_0. \tag{III.8.7}$$

In addition, we have

Proposition III.23 (see [**Si**, Thm. 1.16]). *Let $a, b \in l_0$. Then*

$$\Phi(a) = \Phi(a^*),$$

$$\Big(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N}\Big) \Longrightarrow \Phi(a) \leq \Phi(b).$$

It follows from Proposition III.23 that, for any $a, b \in l_0$,

$$\Big(a_n^* \leq b_n^* \quad \forall n \in \mathbb{N}_0\Big) \Longrightarrow \Phi(a) \leq \Phi(b).$$

In the sequel, we denote by l_f^+ the positive cone of l_f consisting of non-increasing sequences of non-negative numbers with finite supports.

Lemma III.18 ([**GK**, Lem. 3.2]). *Let $\Phi : l_f^+ \to [0, \infty)$ be a function such that*

$$\Phi(a) = 0 \Longrightarrow a = (0, 0, \ldots), \tag{III.8.8}$$

$$\Phi(\lambda a) = \lambda\Phi(a) \qquad \forall a \in l_f^+ \ \forall \lambda \geq 0, \tag{III.8.9}$$

$$\Phi(a+b) \leq \Phi(a) + \Phi(b) \qquad \forall a, b \in l_f^+, \tag{III.8.10}$$

$$\Big(\sigma_N(a) \leq \sigma_N(b) \quad \forall N \in \mathbb{N}\Big) \Longrightarrow \Phi(a) \leq \Phi(b). \tag{III.8.11}$$

Then Φ can be uniquely extend into a symmetric norm on l_f by letting

$$\Phi(a) := \Phi(a^*) \qquad \forall a \in l_f.$$

Finally, let $\Phi' : l_f \to [0, \infty)$ be the function defined by

$$\Phi'(a) := \sup\Big\{\Big|\sum_{n \geq 0} a_n b_n\Big|; \ b \in l_f, \ \Phi(b) \leq 1\Big\} \tag{III.8.12}$$

This is a norm on l_f called the *dual norm* of Φ. Using (III.1) and the fact that Φ is symmetric, we can check that

$$\Phi'(a) = \sup\Big\{\sum_{n \geq 0} a_n^* b_n; \ b \in l_f^+, \ \Phi(b) \leq 1\Big\}, \tag{III.8.13}$$

from which it follows that Φ' is a symmetric norm. It also implies that

$$\sum_{n \geq 0} a_n^* b_n^* \leq\leq \Phi'(a)\Phi(b) \qquad \forall a, b \in l_0. \tag{III.8.14}$$

Lemma III.19 ([**GK**, Thm. 1.11]). *The dual norm of Φ' is equal to Φ, i.e., $(\Phi')' = \Phi$.*

Remark III.21. Two norms Φ and Ψ on l_f are equivalent when there exists $c > 0$ such that

$$c^{-1}\Phi(a) \leq \Psi(a) \leq c\Phi(a) \qquad \forall a \in l_f. \tag{III.8.15}$$

It is not hard to see that Φ and Ψ are equivalent if and only if their dual norms are equivalent.

EXAMPLE III.22. Let $p \in [1,\infty]$ and let $p' \in [1,\infty]$ be such that $\frac{1}{p}+\frac{1}{p'}=1$. Then the dual norm of Φ_p is equal to $\Phi_{p'}$. This follows from the following simple facts:

- The Hölder inequality,
$$\left|\sum_{n\geq 0} a_n b_n\right| \leq \Phi_{p'}(a)\Phi_p(b) \qquad \forall a,b \in l_f.$$
- If $p > 1$, the Hölder inequality is an equality if $b_n = \frac{\overline{a_n}}{|a_n|}|a_n|^{\frac{p'}{p}} = \frac{\overline{a_n}}{|a_n|}|a_n|^{p'-1}$ when $a_n \neq 0$ and $b_n = 0$ otherwise.
- If $p = 1$ the the Hölder inequality is an equality if $b_n = \frac{a_{n_0}}{a_{n_0}}$ for $n = n_0$ and $b_n = 0$ for $n \neq n_0$, where n_0 is such that $|a_{n_0}| = \Phi_\infty(a)$.

III.9. Banach ideals associated to symmetric norms

Let Φ be a symmetric norm on l_f. We shall also denote by Φ its extension to l_0 given by (III.4).

For any operator $T \in \mathcal{K}$, the sequence of characteristic values $\mu(T) := (\mu_n(T))_{n\geq 0}$ is an element if l_0. Therefore, we can set
$$\|T\|_\Phi := \Phi(\mu(T)).$$
We then define
$$\mathcal{I}_\Phi := \left\{T \in \mathcal{K};\ \|T\|_\Phi < \infty\right\}.$$
For $T \in \mathcal{K}$ and $N \in \mathbb{N}$ we define
$$\mu^N(T) := (\mu_0(T), \ldots, \mu_{N-1}(T), 0, 0, \ldots) \in l_f.$$
Then by (III.4) we have

(III.9.1) $$\|T\|_\Phi = \lim_{N\to\infty} \Phi(\mu^N(T)) = \sup_{N\geq 1} \Phi(\mu^N(T)).$$

In addition, we set
$$\sigma_N(T) := \sum_{n<N} \mu_n(T) = \sigma_N(\mu(T)).$$
For $T \in \mathcal{L}(\mathcal{H})$ and $N \in \mathbb{N}$ we set

(III.9.2) $$\sigma_N(T) = \sum_{n<N} \mu_n(T).$$

Using the properties (III.1) and (III.5) of characteristic values we get

(III.9.3) $$\sigma_N(cT) = |c|\sigma_N(T) \qquad \forall c \in \mathbb{C},$$

(III.9.4) $$\sigma_N(ATB) \leq \|A\|\sigma_N(T)\|B\| \qquad \forall A,B \in \mathcal{L}(\mathcal{H}).$$

As $\mu_n(T) \leq \mu_0(T) = \|T\|$ for all $n \in \mathbb{N}$, we also see that

(III.9.5) $$\|T\| \leq \sigma_N(T) \leq N\|T\|.$$

In the sequel if E is a closed subspace of $\mathcal{H}$ we denote by Π_E the orthogonal projection onto E.

LEMMA III.20. *Let $T \in \mathcal{K}$. For any $N \in \mathbb{N}$, we have*

$$\sigma_N(T) = \sup\{\|T\Pi_E\|_1;\ \dim E = N\}, \tag{III.9.6}$$

$$= \sup\{|\operatorname{Trace}(T\Pi_E)|;\ \dim E = N\} \quad \textit{(if } T \textit{ is positive).} \tag{III.9.7}$$

PROOF. It follows from (III.4) that $\sigma_N(T) = \sigma_N(|T|)$. Moreover, if E is a closed subspace of $\mathcal{H}$, then $||T|\Pi_E| = |T\Pi_E|$, for $||T|\Pi_E|$ is a positive operator such that

$$||T|\Pi_E|^2 = (|T|\Pi_E)^*(|T|\Pi_E) = \Pi_E|T|^2\Pi_E = \Pi_E T^*T\Pi_E = (T\Pi_E)^*(T\Pi_E).$$

Therefore, upon replacing T by $|T|$ we may assume T positive.

Notice also that if E is a closed subspace of $\mathcal{H}$, then by (III.12)

$$|\operatorname{Trace}(T\Pi_E)| \leq \|T\Pi_E\|_1,$$

and hence

$$\sup\{|\operatorname{Trace}(T\Pi_E)|;\ \dim E = N\} \leq \sup\{\|T\Pi_E\|_1;\ \dim E = N\}. \tag{III.9.8}$$

Let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be an orthonormal family such that $T\xi_n = \mu_n(T)$ for all $n \in \mathbb{N}_0$. Then by Proposition (III.5) we have

$$T = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*,$$

where the series converges in norm. Let E_N be N-dimensional subspace spanned by $\xi_0, \ldots, \xi_{N-1}$; this is a subspace of dimension N. Then $\Pi_{E_N} = \sum_{n<N} \xi_n \otimes \xi_n^*$, and hence

$$T\Pi_{E_N} = \sum_{n<N} \mu_n(T)\xi_n \otimes \xi_n^*.$$

Thus the $(n+1)$'th eigenvalue of $T\Pi_N$ counted with multiplicity is equal to $\mu_n(T)$ if $n < N$ and is zero if $N \geq 0$. Therefore, using (III.16) we get

$$\operatorname{Trace}(T\Pi_{E_N}) = \sum_{n<N} \mu_n(T) = \sigma_n(T).$$

Since $\dim E_N = N$, it follows that

$$\sigma_N(T) \leq \sup\{|\operatorname{Trace}(T\Pi_E)|;\ \dim E = N\}. \tag{III.9.9}$$

Let E be an N-dimensional subspace of $\mathcal{H}$. Then $T\Pi_E$ has rank $\leq N$, and so using Proposition III.1 we see that $\mu_n(T\Pi_E) = 0$ for $n \geq N$. Thus,

$$\|T\Pi_E\|_1 = \sum_{n\geq 0} \mu_n(T\Pi_E) = \sum_{n<N} \mu_n(T\Pi_E). \tag{III.9.10}$$

Thanks to (III.5) and the fact that Π_E is an orthogonal projection we have

$$\mu_n(T\Pi_E) \leq \mu_n(T)\|\Pi_E\| \leq \mu_n(T).$$

Combining this with (III.10) we get

$$\|T\Pi_E\|_1 \leq \sum_{n<N} \mu_n(T) = \sigma_N(T),$$

and hence

$$\sup\{\|T\Pi_E\|_1;\ \dim E = N\} \leq \sigma_N(T).$$

Combining this with (III.8) and (III.9) proves the lemma. $\square$

Notice that for every subspace E of dimension N the function $T \to \|T\Pi_E\|_1$ is a semi-norm on $\mathcal{K}$, so as a supremum of all such semi-norms σ_N is a semi-norm on $\mathcal{K}$. In particular, we have:

LEMMA III.21. *Let $N \in \mathbb{N}$. Then*

$$\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T) \qquad \forall S, T \in \mathcal{K}. \tag{III.9.11}$$

Granted this lemma we shall prove:

LEMMA III.22. *The following hold.*

(1) *Let $T \in \mathcal{K}$. Then*

$$\|T\|_\Phi = 0 \Longrightarrow T = 0, \tag{III.9.12}$$

$$\|\lambda T\|_\Phi = |\lambda| \|T\|_\Phi \qquad \forall \lambda \in \mathbb{C}, \tag{III.9.13}$$

$$\|ATB\|_\Phi \leq \|A\| \|T\|_\Phi \|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{III.9.14}$$

(2) *Let $S, T \in \mathcal{K}$. Then*

$$\|S+T\|_\Phi \leq \|S\|_\Phi + \|T\|_\Phi.$$

(3) *If $\Phi(1,0,0,\ldots) = 1$, then*

$$\|T\|_\Phi = \|T\| \qquad \forall T \in \mathcal{R}_1, \tag{III.9.15}$$

$$\|T\| \leq \|T\|_\Phi \qquad \forall T \in \mathcal{K}. \tag{III.9.16}$$

PROOF. The implication (III.12) is due to (III.5) and the fact that $\mu_0(T) = \|T\|$. We obtain (III.13) by using (III.1) and (III.6). The inequality (III.14) follows by combining (III.5) and (III.8).

Let $S, T \in \mathcal{K}$. Then (III.11) shows that $\sigma_N(\mu(S+T)) \leq \sigma_N(\mu(S) + \mu(T))$ for all $N \in \mathbb{N}$, so using Proposition III.23 and (III.7) we get

$$\|S+T\|_\Phi = \Phi(\mu(S+T)) \leq \Phi(\mu(S)+\mu(T)) \leq \Phi(\mu(S)) + \Phi(\mu(T)) = \|S\|_\Phi + \|T\|_\Phi.$$

Suppose now that $\Phi(1,0,0,\ldots) = 1$ and let $T \in \mathcal{K}$. As $\mu^0(T) = \|T\|(1,0,0,\ldots)$ we see that $\Phi(\mu^0(T)) = \|T\|\Phi(1,0,0,\ldots) = \|T\|$. Since $\mu_n(T) \geq \mu_n^0(T)$ for all $n \in \mathbb{N}_0$, using (III.8) we see that $\|T\|_\Phi \geq \Phi(\mu^0(T)) = \|T\|$. Moreover, if $\operatorname{rk} T = 1$ then $\mu(T) = \mu^0(T)$, and hence $\|T\|_\Phi = \Phi(\mu^0(T)) = \|T\|$. The lemma is proved. ☐

PROPOSITION III.24. *$\mathcal{I}_\Phi$ is a Banach ideal for $\|.\|_\Phi$, i.e., $\mathcal{I}_\Phi$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_\Phi$ is a Banach norm on $\mathcal{I}_\Phi$ satisfying (III.1).*

PROOF. It follows from Lemma III.22 that $\mathcal{I}_\Phi$ is a a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_\Phi$ is a norm on $\mathcal{I}_\Phi$ satisfying (III.1). It remains to check that $\mathcal{I}_\Phi$ is complete for the norm $\|.\|_\Phi$.

Let $(T_n)_{n\geq 0}$ be a Cauchy sequence in $(\mathcal{I}, \|.\|_\mathcal{I})$ and let us show that it converges in $(\mathcal{I}, \|.\|_\mathcal{I})$. Then (III.16)implies that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{K}$, hence converges in $\mathcal{K}$ to some operator T.

Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|T_p - T_q\|_\Phi < \epsilon$ when p and p are greater than n_0. Let $N \in \mathbb{N}$ and denote by $e^N = (e^N)_{n\geq 0}$ the sequence such that $e_n^N = 1$ for $n < N$ and $e_n^N = 0$ for $n \geq N$. Let p and q be integers greater than n_0. It follows from (III.8) that, for all $n \in \mathbb{N}_0$,

$$\mu_n^N(T - T_p)) \leq \mu_n^N(T_p - T_q) + \|T - T_q\| e_n^N.$$

Therefore, using (III.7) and (III.8) we see that

$$\Phi(\mu^N(T-T_p) \leq \Phi(\mu^N(T_p-T_q)) + \Phi(\|T-T_q\|e_n^N)$$
$$\leq \|T_p-T_q\|_\Phi + \|T-T_q\|\Phi(e^N) \leq \epsilon + \|T-T_q\|\Phi(e^N).$$

Letting $q \to \infty$ shows that

$$\Phi(\mu^N(T-T_p) \leq \epsilon + \|T-T_q\|\Phi(e^N) \qquad \forall N \in \mathbb{N}.$$

Combining this with (III.4) we get

$$\|T-T_p\|_\Phi \leq \epsilon \qquad \forall p > n_0. \tag{III.9.17}$$

This implies that T is contained in $\mathcal{I}_\Phi$ and the sequence $(T_n)_{n\geq 0}$ converges to T in $(\mathcal{I}_\Phi, \|.\|_\Phi)$, proving that $(\mathcal{I}_\Phi, \|.\|_\Phi)$ is a Banach space. The proof is complete. □

Remark III.23. If Ψ is another symmetric norm on l_f then $\mathcal{I}_\Phi = \mathcal{I}_\Psi$ if and only if Φ and Ψ are equivalent in the sense of (III.15). Furthermore, it is immediate that in that case the norms $\|.\|_\Phi$ and $\|.\|_\Psi$ are equivalent on $\mathcal{I}_\Phi = \mathcal{I}_\Psi$.

Lemma III.23. *Let $T \in \mathcal{I}_\Phi$. Then T is contained in $\mathcal{I}_\Phi^0$ if and only if*

$$\lim_{N\to\infty} \Phi(\mu_N(T), \mu_{N+1}(T), \ldots) = 0. \tag{III.9.18}$$

Proof. Lemma III.15 says that T is contained in $\mathcal{I}_\Phi^0$ if and only if any Schmidt series (III.5) for T converges to T in $\mathcal{I}_\Phi$. Let $T = U|T|$ be the polar decomposition of T and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. It immediately follows from (III.7) that

$$\Big\| \sum_{n\geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^* \Big\|_\Phi = \Phi(\mu_N(T), \mu_{N+1}(T), \ldots).$$

Thus T is contained in $\mathcal{I}_\Phi^0$ if and only if (III.18) holds. □

Combining this lemma with Proposition III.21 we obtain:

Proposition III.25. *The following are equivalent.*

1. *The Banach ideal $\mathcal{I}_\Phi$ is separable.*
2. *The finite-rank operators are dense in $\mathcal{I}_\Phi$, i.e., $\mathcal{I}_\Phi^0 = \mathcal{I}_\Phi$.*
3. *For any $a \in l_0$,*

$$\Phi(a) < \infty \Longrightarrow \lim_{N\to\infty} \Phi(a_N, a_{N+1}, \ldots) = 0. \tag{III.9.19}$$

Proposition III.26. *Let $\mathcal{I}$ be a Banach ideal with norm $\|.\|_\mathcal{I}$. Then*

1. *There exists a unique symmetric norm on l_f such that*

$$\mathcal{I} \subset \mathcal{I}_\Phi \qquad \textit{and} \qquad \|T\|_\mathcal{I} = \|T\|_\Phi \quad \forall T \in \mathcal{R}_\infty. \tag{III.9.20}$$

2. *The Banach ideals $\mathcal{I}^0$ and $\mathcal{I}_\Phi^0$ coincide.*

Proof. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For any $a \in l_0$ set

$$T_a = \sum_{n\geq 0} a_n \xi_n \otimes \xi_n^*.$$

As $\lim_{n\to\infty} a_n = 0$ the above series converges in $\mathcal{K}$, i.e., T_a is a compact operator. Observe also that $\mu_n(T_a) = a_n^*$ $\forall n \in \mathbb{N}_0$. In addition, if $T \in \mathcal{K}$, then $\mu(T_{\mu(T)}) = \mu(T)$.

Let $\Phi : l_f \rightarrow [0, \infty)$ be the function function defined by

$$\Phi(a) = \|T_a\|_{\mathcal{I}} \qquad \forall a \in l_f.$$

It is not hard to check that Φ is a norm on l_f. Moreover, as $\mu(T_a) = a^* = \mu(T_{a^*})$, using Lemma III.14 we see that $\|T_a\|_{\mathcal{I}} = \|T_{a^*}\|_{\mathcal{I}}$, i.e., $\Phi(a) = \Phi(a^*)$. Thus Φ is a symmetric norm on l_f. Let $\mathcal{I}_\Phi$ be the associated Banach ideal with norm $\|.\|_\Phi$.

Let $T \in \mathcal{I}$ and let $N \in \mathbb{N}$. As $\mu_n(T_{\mu^N(T)}) = \mu_n^N(T) \leq \mu_n(T)$, it follows from Lemma III.14 that

$$\Phi(\mu^N(T)) = \|T_{\mu^N(T)}\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}.$$

Thus,

$$\|T\|_\Phi = \sup_{N \geq 1} \Phi(\mu^N(T)) \leq \|T\|_{\mathcal{I}} < \infty,$$

that is, the operator T is contained in $\mathcal{I}_\Phi$. If in addition T has finite-rank then, as $\mu(T) = \mu(T_{\mu(T)})$, Lemma III.14 insures us that

$$\|T\|_{\mathcal{I}} = \|T_{\mu(T)}\|_{\mathcal{I}} = \Phi(\mu(T)) = \|T\|_\Phi.$$

Therefore $\|.\|_{\mathcal{I}}$ and $\|.\|_\Phi$ agree on finite-rank operators.

Let Ψ be another symmetric norm on l_f such that $\mathcal{I} \subset \mathcal{I}_\Psi$ and $\|.\|_\Psi = \|.\|_{\mathcal{I}}$ on $\mathcal{R}_\infty$. Let $a \in l_f$. As Ψ is symmetric and $\mu(T_a) = a^*$, we have

$$\Psi(a) = \Psi(a^*) = \Psi(\mu(T_a)) = \|T_a\|_\Psi = \|T_a\|_{\mathcal{I}} = \Phi(a).$$

Therefore Ψ and Φ agrees, so Φ is the unique symmetric norm on l_f satisfying (III.20).

Let $T \in \mathcal{K}$. As a Schmidt series for T is a series of finite-rank operators, it follows from (III.20) that a Schmidt series for T satisfies Cauchy's criterion for $\|.\|_{\mathcal{I}}$ if and and only it satisfies it for $\|.\|_\Phi$. Using Lemma III.15 we then deduce that T is contained in $\mathcal{I}^0$ if and only if it is contained in $\mathcal{I}^0_\Phi$. Thus, as sets, $\mathcal{I}^0$ and $\mathcal{I}^0_\Phi$ agree.

Let T be in $\mathcal{I}^0 = \mathcal{I}^0_\Phi$ and let $\sum_{n \geq 0} \mu_n(T)(U\eta_n) \otimes \eta_n^*$ be a Schmidt series for T. As this series converges to T both in $\mathcal{I}$ and in $\mathcal{I}_\Phi$, using (III.20) we get

$$\|T\|_{\mathcal{I}} = \lim_{N \rightarrow \infty} \Big\| \sum_{n<N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \Big\|_{\mathcal{I}} = \lim_{N \rightarrow \infty} \Big\| \sum_{n<N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \Big\|_\Phi = \|T\|_\Phi.$$

Thus $\|.\|_{\mathcal{I}}$ and $\|.\|_\Phi$ agrees on $\mathcal{I}^0$. This proves that the Banach ideals $\mathcal{I}^0$ and $\mathcal{I}^0_\Phi$ coincide. The proof is complete. □

Combining Proposition III.25 and Proposition III.26 we obtain:

Proposition III.27. *A Banach ideal $\mathcal{I}$ is separable if and only if there exists a symmetric norm Φ on l_f such that $\mathcal{I}$ coincides with the Banach ideal $\mathcal{I}^0_\Phi$.*

Next, let us denote by Φ' the dual symmetric norm of Φ as defined in (III.12). We can relate the Banach ideal $\mathcal{I}_{\Phi'}$ to the dual of $\mathcal{I}_\Phi$ as follows.

Lemma III.24 (Horn Inequality; see [**Si**, Thm. 1.15]). *Let $S, T \in \mathcal{K}$. Then*

$$\sum_{n<N} \mu_n(ST) \leq \sum_{n<N} \mu_n(S)\mu_n(T) \qquad \forall N \in \mathbb{N}.$$

Proposition III.28. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{\Phi'} \|T\|_\Phi.$$

(2) *Let* $S \in \mathcal{I}_{\Phi'}$ *and* $T \in \mathcal{I}_{\Phi}$. *Then the operator* ST *is trace-class and*

$$|\operatorname{Trace}(ST)| \leq \|S\|_{\Phi'}\|T\|_{\Phi}. \tag{III.9.21}$$

(3) *For all* $S \in \mathcal{K}$, *we have*

$$\|S\|_{\Phi'} = \sup_{\substack{\|T\|_{\Phi}=1 \\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)|. \tag{III.9.22}$$

Proof. Let $S, T \in \mathcal{K}$. Then using Horn's inequality and (III.14) we see that, for any $N \in \mathbb{N}$, we have

$$\sum_{n<N} \mu_n(ST) \leq \sum_{n\geq 0} \mu_n^N(S)\mu^N(T) \leq \Phi'(\mu^N(S))\Phi(\mu^N(T)) \leq \|S\|_{\Phi'}\|T\|_{\Phi}.$$

Thus,

$$\|ST\|_1 = \sum_{n\geq 0} \mu_n(ST) \leq \|S\|_{\Phi'}\|T\|_{\Phi}.$$

Therefore, if $S \in \mathcal{I}_{\Phi'}$ and $T \in \mathcal{I}_{\Phi}$, then ST is trace-class and, using (III.12), we get

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_{\Phi'}\|T\|_{\Phi}. \tag{III.9.23}$$

Let $S \in \mathcal{K}$. Then (III.23) implies that

$$\|S\|_{\Phi'} \geq \sup_{\substack{\|T\|_{\Phi}=1 \\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)|. \tag{III.9.24}$$

Let $A \in (0, \|S\|_{\Phi})$. In view of (III.1) we can find $N \in \mathbb{N}$ large enough such that $A < \Phi'(\mu^N(S))$. Using (III.14) we see that there exists a sequence $b = (b_n)_{n\geq 0}$ in l_f^+ with same support as μ^N such that $\Phi'(b) = 1$ and

$$A < \sum_{n\geq 0} \mu_n^N(S)b_n \leq \sum_{n\geq 0} \mu_n(S)b_n. \tag{III.9.25}$$

Let $S = U|S|$ be the polar decomposition of S and let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be an orthonormal family such that $|S|\xi_n = \mu_n(S)\xi_n$ $\forall n \in \mathbb{N}_0$. Set

$$T = \sum_{n\geq 0} b_n(\xi_n \otimes \xi_n^*)U^*.$$

The operator T has finite rank, since the support of b is finite.

By Proposition II.3, the operator U^*U is the orthogonal projection onto $(\ker S)^\perp = (\ker |S|)^\perp$, so $U^*U\xi_n = \xi_n$ whenever $\mu_n(S) \neq 0$. Therefore, we can check that

$$T^*T = \sum_{n\geq 0} b_n^2 \xi_n \otimes \xi_n^* \quad \text{and} \quad |T| = \sum_{n\geq 0} b_n \xi_n \otimes \xi_n^*$$

Using the min-max principle we then deduce that $\mu_n(T) = b_n$ for all $n \in \mathbb{N}_0$. Thus,

$$\|T\|_{\Phi} = \Phi(\mu(T)) = \Phi(b) = 1.$$

We also have

$$ST = \Big(\sum_{n\geq 0} \mu_n(S)U(\xi_n \otimes \xi_n^*)\Big)\Big(\sum_{n\geq 0} b_n(\xi_n \otimes \xi_n^*)U^*\Big) = \sum_{n\geq 0} \mu_n(S)b_n U(\xi_n \otimes \xi_n^*)U^*,$$

Thus $\operatorname{Trace}(ST)$ is equal to

$$\sum_{n\geq 0} \mu_n(S)b_n \operatorname{Trace}\left(U(\xi_n \otimes \xi_n^*)U^*\right) = \sum_{n\geq 0} \mu_n(S)b_n \operatorname{Trace}\left(U^*U(\xi_n \otimes \xi_n^*)\right) = \sum_{n\geq 0} \mu_n(S)b_n.$$

In view of (III.25) this implies that $\operatorname{Trace}(ST) > A$. Since $\|T\|_\Phi = 1$ we deduce that

$$A < \sup_{\substack{\|T\|_\Phi=1\\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \qquad \forall A \in (0, \|S\|_\Phi).$$

Combining this with (III.24) yields (III.22). The proof is complete. $\square$

Recall that by Proposition III.9 we have an isometric isomorphism from $\mathcal{L}(\mathcal{H})$ onto $(\mathcal{L}^1)'$ given by

(III.9.26)

$$\mathcal{L}(\mathcal{H}) \ni S \longrightarrow (S,.) \in (\mathcal{L}^1)', \qquad (S,.) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S,T) := \operatorname{Trace}(ST).$$

It follows from Proposition III.28 that we also have linear map,

(III.9.27)

$$\mathcal{I}_{\Phi'} \ni S \longrightarrow (S,.) \in (\mathcal{I}_\Phi^0)', \qquad (S,.) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S,T) := \operatorname{Trace}(ST).$$

and the density of $\mathcal{L}^1$ in $\mathcal{I}_\Phi^0$ show that if $S \in \mathcal{I}_{\Phi'}$ then $(S,.)$ uniquely extends to a continuous linear map on $\mathcal{I}_\Phi^0$, which we shall continue to denote $(S,.)$.

Proposition III.29. *If $\mathcal{I}_\Phi \supsetneq \mathcal{L}^1$, then (III.27) yields an isometric isomorphism,*

$$\mathcal{I}_{\Phi'} \simeq (\mathcal{I}_\Phi^0)'.$$

Proof. Let us denote by $\mathcal{T}_\Phi$ the linear map (III.27). It follows from (III.21) and (III.22) that, for all $S \in \mathcal{I}_{\Phi'}$,

$$\|S\|_{\Phi'} = sup_{\substack{\|T\|_\Phi=1\\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \leq \sup_{\substack{\|T\|_\Phi=1\\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \leq \|S\|_{\Phi'}.$$

Therefore $\mathcal{T}_\Phi$ is an isometry, so it follows from Lemma I.1 that for proving that $\mathcal{T}_\Phi$ is an isomorphism it is enough to show that it is onto.

Consider the following subspace of $\mathcal{L}(\mathcal{H})$,

$$\mathcal{I} := \mathcal{T}^{-1}((\mathcal{I}_\Phi^0)') = \left\{ S \in \mathcal{L}(\mathcal{H});\ \sup_{T\in\mathcal{R}_\infty\setminus 0} \frac{|\operatorname{Trace}(ST)|}{\|T\|_\Phi} < \infty \right\}.$$

Let $S \in \mathcal{I}$, let $A, B \in \mathcal{L}(\mathcal{H})$ and $C := \sup_{T\in\mathcal{R}_\infty\setminus 0} \frac{|\operatorname{Trace}(ST)|}{\|T\|_\Phi}$. Then, for all $T \in \mathcal{R}_\infty$,

$$|\operatorname{Trace}(ASBT)| = |\operatorname{Trace}(SBTA)| \leq C\|BTA\|_\Phi \leq C\|B\|\|T\|_\Phi\|A\|,$$

hence ASB is contained in $\mathcal{I}$. This shows that $\mathcal{I}$ is a two-sided ideal.

Suppose that $\mathcal{I}$ is not contained in $\mathcal{K}$. As $\mathcal{I}$ is a two-sided ideal, Proposition III.18 then insures us that $\mathcal{I} = \mathcal{L}(\mathcal{H})$ and $\mathcal{K} = \mathcal{I} \cap \mathcal{K}$. Since (III.22) shows that $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, we see that $\mathcal{I}_{\Phi'} = \mathcal{K}$. Observe that $\mathcal{K}$ is the Banach ideal $\mathcal{I}_{\Phi_\infty}$ associated to the norm Φ_∞ in (III.2). Using Remark III.23 we see that Φ' and Φ_∞ are equivalent norms on l_f, so by Remark III.21 their dual norms. By Lemma III.19 the dual norm of Φ' is Φ and, as shown in Example III.22, the dual norm of Φ_∞ is the norm Φ_1. Combining this with Remark III.23, we see that $\mathcal{I}_\Phi = \mathcal{I}_{\Phi_1} = \mathcal{L}^1$. This contradicts the assumption that $\mathcal{I}_\Phi$ does not coincide with $\mathcal{L}^1$, so $\mathcal{I}$ must be contained in $\mathcal{K}$. As $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, this proves that $\mathcal{I} = \mathcal{I}_{\Phi'}$.

Let $\varphi \in (\mathcal{I}_\Phi^0)'$. By Proposition III.22 the inclusion $\mathcal{L}^1 \subset \mathcal{L}_\Phi^0$ is continuous, so φ induces a continuous linear map on $\mathcal{L}^1$. The isomorphism (III.26) insures us that there exists $S \in \mathcal{L}(\mathcal{H})$ such that

$$\langle \varphi, T \rangle = \operatorname{Trace}(ST) \qquad \forall T \in \mathcal{L}^1. \tag{III.9.28}$$

Then, for all $T \in \mathcal{R}_\infty$, we have

$$|\operatorname{Trace}(ST)| = |\langle \varphi, T\rangle| \leq \|\varphi\|_{\mathcal{I}'_\Phi} \|T\|_\Phi.$$

This shows that S is contained in $\mathcal{I} = \mathcal{I}_{\Phi'}$. Observe that (III.28) shows that φ and $\mathcal{T}_\Phi$ agree on $\mathcal{L}^1$. In particular, they agree on finite-rank operators. As finite-rank operators are dense in $\mathcal{I}^0_\Phi$, it follows that φ and $\mathcal{T}_\Phi$ agrees on all $\mathcal{I}^0_\Phi$, i.e., $\mathcal{T}_\Phi(S) = \varphi$. This proves that $\mathcal{T}_\Phi$ is onto, completing the proof. □

III.10. Schatten Ideals

Let $p \in [0, \infty]$. The Schatten ideal $\mathcal{L}^p$ is the Banach ideal $\mathcal{L}_{\Phi_p}$ associated to the p-norm Φ_p. Thus, if for any $T \in \mathcal{K}$, we set

$$\|T\|_p := \Phi_p\left((\mu_n(T))_{n\geq 0}\right) = \left(\sum_{n\geq 0} \mu_n(T)^p\right)^{\frac{1}{p}},$$

then

$$\mathcal{L}^p = \left\{T \in \mathcal{K}; \|T\|_p < \infty\right\}.$$

For $p = 1$ (resp. $p = 2$) we recover the Banach ideal of trace-class operators (resp. Hilbert-Schmidt operators). As alluded to in the proof of Proposition III.29, for $p = \infty$ the Banach ideal $\mathcal{L}^\infty = \mathcal{I}_{\Phi_\infty}$ is the whole Banach ideal of compact operators.

As shown in Example III.3 we have $\mu_n(|T|^p) = \mu_n(T)^p$ $\forall n \in \mathbb{N}$, so we have

$$\sum_{n\geq 0} \mu_n(T)^p = \operatorname{Trace}|T|^p \qquad \forall T \in \mathcal{L}(\mathcal{H}).$$

Thus,

$$T \in \mathcal{L}^p \Longleftrightarrow \operatorname{Trace}|T|^p < \infty \Longleftrightarrow |T|^p \in \mathcal{L}^1.$$

Proposition III.30. *Let $q \in (p, \infty)$. Then*

$$\|T\|_q \leq \|T\|_p \qquad \forall T \in \mathcal{K}.$$

In particular, there is a continuous inclusion,

$$\mathcal{L}^q \subset \mathcal{L}^p.$$

Proof. Let $T \in \mathcal{K}$. Observe that, for all $n \in \mathbb{N}_0$,

$$\mu_n(T)^q = \mu_n(T)^{q-p}.\mu_n(T)^p \leq (\|T\|_p)^{q-p}.\mu_n(T)^p.$$

Therefore, we get

$$\|T\|_q = \left(\sum_{n\geq 0}\mu_n(T)^q\right)^{\frac{1}{q}} \leq (\|T\|_p)^{\frac{q-p}{q}}\left(\sum_{n\geq 0}\mu_n(T)^p\right)^{\frac{1}{q}} = (\|T\|_p)^{\frac{q-p}{q}}(\|T\|_p)^{\frac{p}{q}} = \|T\|_p,$$

proving the lemma. □

It can be easily seen that, for $p < \infty$, the symmetric norm Φ_p satisfies the condition (III.19). Therefore, Proposition III.25 gives

Proposition III.31. *The Schatten ideal $\mathcal{L}^p$ is separable and the finite-rank operators are dense in $\mathcal{L}^p$.*

Assume $p > 1$ and let $p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual norm of Φ_p is the p'-norm $\Phi_{p'}$. This follows from the Hölder inequality,

$$\left|\sum_{n\geq 0} a_n b_n\right| \leq \left(\sum_{n\geq 0} |a_n|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n\geq 0} |b_n|^{p}\right)^{\frac{1}{p}} \qquad \forall a, b \in l_f,$$

and the fact that we actually have an equality when $b_n = \frac{\overline{a_n}}{|a_n|}|a_n|^{\frac{p'}{p}}$. Therefore, using Proposition III.28 and Proposition III.29 we obtain:

Proposition III.32. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{p'}\|T\|_p.$$

(2) *If $S \in \mathcal{L}^{p'}$ and $T \in \mathcal{L}^p$, then the operator ST is trace-class and*

$$|\operatorname{Trace}(ST)| \leq \|S\|_q\|T\|_p.$$

(3) *The linear map (III.27) gives rise to an isometric isomorphism,*

$$\mathcal{L}^{p'} \simeq (\mathcal{L}^p)'.$$

The Horn inequality admits the following generalization (see [**Si**, Thm. 1.15]). Let $r \in [1, \infty)$ and $N \in \mathbb{N}$. Then

$$\sum_{n<N} \mu_n(ST)^r \leq \sum_{n<N} \mu_n(S)^r \mu_n(T)^r \qquad \forall S, T \in \mathcal{K}.$$

Using this generalization we can show that, if $\frac{1}{p_1} + \ldots + \frac{1}{p_k} = \frac{1}{r}$, then

$$\|T_1T_2\cdots T_k\|_1 \leq \|T_1\|_{p_1}\|T_2\|_{p_2}\cdots\|T_1\|_{p_k} \qquad \forall T_j \in \mathcal{K}.$$

In particular, if for every $j = 1, \ldots, k$ the operator T_j is in $\mathcal{L}^{p_j}$, then $T_1T_2\cdots T_k$ is a trace-class operator and we have

$$|\operatorname{Trace}(T_1T_2\cdots T_k)| \leq \|T_1\|_{p_1}\|T_2\|_{p_2}\cdots\|T_1\|_{p_k}.$$

III.11. Banach ideals associated to divergent series

Following [**GK**] we can produce a large class of non-separable ideals as follows.

Let $\pi = (\pi_n)_{n\geq 0}$ be a non-increasing sequence of positive real numbers satisfying the following two conditions:

$$\lim_{n\to\infty} \pi_n = 0 \qquad \text{and} \qquad \sum_{n\geq 0} \pi_n = \infty. \tag{III.11.1}$$

Examples of sequences satisfying all these conditions are provided by the sequences

$$\pi^{(p)} := ((n+1)^{\frac{1}{p}})_{n\geq 0}, \qquad p \geq 1. \tag{III.11.2}$$

Using Lemma III.18 it is not difficult to check that we define a symmetric norm on l_f by letting

$$\Phi_\pi(a) := \sup_{N\geq 1} \frac{\sigma_N(a^*)}{\sigma_N(\pi)}.$$

We denote by $\mathcal{I}_{\Phi_\pi}$ the associated Banach ideal. In particular,

$$\mathcal{I}_{\Phi_\pi} = \Big\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\sigma_N(\pi))\Big\}. \tag{III.11.3}$$

Lemma III.25. *Let $a=(a_n)_{n\geq 0}$ be a non-increasing sequence in l_0 such that $\Phi_\pi(a)<\infty$. Then*

$$\lim_{N\to\infty}\Phi(a_N,a_{N+1},\ldots)=\limsup_{N\to\infty}\frac{\sigma_N(a)}{\sigma_N(\pi)}.$$

Proof. For $N\in\mathbb{N}$ set $a^N=a_N,a_{N+1},\ldots)\in l_0$. Let $n\in\mathbb{N}_0$. As the sequence $(a_n)_{n\geq 0}$ is non-decreasing, we have

$$\sigma_n(a^N)=\sum_{j<N}a_{j+N}=\sigma_{n+N}(a)-\sigma_n\geq\sigma_n(a)-\sigma_N(a).$$

Thus,

$$\frac{\sigma_n(a)}{\sigma_n(\pi)}\leq\frac{\sigma_n(a^N)}{\sigma_n(\pi)}+\frac{\sigma_N(a)}{\sigma_n(\pi)}\leq\Phi_\pi(a^N)+\frac{\sigma_N(a)}{\sigma_n(\pi)}.$$

Since (III.1) implies that $\lim_{n\to\infty}\sigma_n(\pi)=\sum_{j\geq 0}a_j=\infty$, we deduce that

$$\limsup_{n\to\infty}\frac{\sigma_n(a)}{\sigma_n(\pi)}\leq\Phi_\pi(a^N)\qquad\forall N\in\mathbb{N}.$$

Thus,

$$\limsup_{n\to\infty}\frac{\sigma_n(a)}{\sigma_n(\pi)}\leq\lim_{N\to\infty}\Phi_\pi(a^N).\tag{III.11.4}$$

Let N and m be positive integers. As the sequence $(a_n)_{n\geq 0}$ is non-decreasing, for any $n\in\mathbb{N}$, we have

$$\sigma_n(a^N)=\sum_{j<n}a_{N+j}\leq\sum_{j<n}a_j=\sigma_n(a).$$

Therefore, for all $n\geq m$, we have

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)}\leq\frac{\sigma_n(a)}{\sigma_n(\pi)}\leq\sup_{p\geq m}\frac{\sigma_p(a)}{\sigma_p(\pi)}.$$

Notice also that, for all $n\leq m-1$,

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)}\leq\frac{na_0^N}{\pi_0}=\frac{m}{\pi_0}a_N.$$

Therefore, we have

$$\Phi_\pi(a^N)=\sup_{n\geq 1}\frac{\sigma_n(a^N)}{\sigma_n(\pi)}\leq\sup\left\{\sup_{p\geq m}\frac{\sigma_p(a)}{\sigma_p(\pi)},\frac{m}{\pi_0}a_N\right\}.$$

Since $\lim_{N\to\infty}a_N=0$, it follows that, ,

$$\lim_{N\to\infty}\Phi_\pi(a^N)\leq\sup_{p\geq m}\frac{\sigma_p(a)}{\sigma_p(\pi)}\qquad\forall m\in\mathbb{N}.$$

Thus,

$$\lim_{N\to\infty}\Phi_\pi(a^N)\leq\lim_{m\to\infty}\sup_{p\geq m}\frac{\sigma_p(a)}{\sigma_p(\pi)}=\limsup_{n\to\infty}\frac{\sigma_n(a)}{\sigma_n(\pi)}.$$

Combining this with (III.4) yields the lemma. □

Proposition III.33. *The Banach ideal $\mathcal{I}_{\Phi_\pi}$ is not separable and*

$$\mathcal{I}^0_{\Phi_\pi}=\left\{T\in\mathcal{K};\sigma_N(T)=\mathrm{o}(\sigma_N(\pi))\right\}.\tag{III.11.5}$$

Proof. It follows from Lemma III.23 and Lemma III.25 that an operator $T \in \mathcal{K}$ is in $\mathcal{I}^0_{\Phi_\pi}$ if and only if $\sigma_N(T) = \mathrm{o}(\sigma_N(\pi))$.

Thanks to (III.1) the sequence $\pi = (\pi_n)_{n\geq 0}$ is contained in l_0. It is immediate that $\Phi_\pi(\pi) = 1$. Moreover, using Lemma III.25, we see that

$$\lim_{N\to\infty} \Phi_\pi(\pi_N, \pi_{N+1}, \ldots) = \limsup_{N\to\infty} \frac{\sigma_N(\pi)}{\sigma_N(\pi)} = 1.$$

Therefore (III.19) does hold, so using Proposition III.25 we see that the Banach ideal $\mathcal{I}_{\Phi_\pi}$ is not separable. The proof is complete. ☐

Proposition III.34. *Suppose that* $\sigma_n(\pi) = \mathrm{O}(n\pi_n)$. *Then*

$$\mathcal{I}_{\Phi_\pi} = \Big\{T \in \mathcal{K}; \sigma_N(T) = \mathrm{O}(\pi_n)\Big\} \quad \textit{and} \quad \mathcal{I}^0_{\Phi_\pi} = \Big\{T \in \mathcal{K}; \sigma_N(T) = \mathrm{o}(\pi_n)\Big\}.$$

Proof. Let $T \in \mathcal{K}$. Let $m \in \mathbb{N}$ and set $a_m = \sup_{n\geq m} \frac{\mu_n(T)}{\pi_n}$. For $n \geq m$ we have $\mu_n(T) = \frac{\mu_n(T)}{\pi_n}\pi_n \leq a_m \pi_n$, so for all $N > m$ we get

$$\sigma_N(T) = \sigma_m(T) + \sum_{m\leq n<N} \mu_n(T) \leq \sigma_m(T) + a_m \sum_{m\leq n<N} \pi_n \leq \sigma_m(T) + a_m \sigma_N(T).$$

Since $\lim_{N\to\infty} \sigma_N(\pi) = \infty$, we deduce that

$$\limsup_{N\geq\infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq a_m \qquad \forall m \in \mathbb{N}.$$

Thus,

$$\limsup_{N\to\infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq \lim_{m\to\infty} a_m = \limsup_{n\to\infty} \frac{\mu_n(T)}{\pi_n}. \tag{III.11.6}$$

Set $C := \sup_{n\geq 1} \frac{\sigma_n(\pi)}{n\pi_n}$. Since the sequence $(\mu_n(T))_{n\geq 0}$ is non-decreasing, for any $n \in \mathbb{N}$, we have $\sigma_n(T) = \sum_{j<n} \mu_j(T) \geq n\mu_n(T)$, and hence

$$\frac{\mu_n(T)}{\pi_n} \leq \frac{\sigma_n(T)}{n\pi_n} = \frac{\sigma_n(\pi)}{n\pi_n} . \frac{\sigma_n(T)}{\sigma_n(\pi)} \leq C \frac{\sigma_n(T)}{\sigma_n(\pi)}.$$

Thus,

$$\limsup_{n\to\infty} \frac{\mu_n(T)}{\pi_n} \leq C \limsup_{n\to\infty} \frac{\sigma_n(T)}{\sigma_n(\pi)}. \tag{III.11.7}$$

Combining (III.6)–(III.7) with (III.3) and (III.5) yields the proposition. ☐

Let Φ'_π be the dual norm of Φ_π as defined in (III.12). This is a symmetric norm on l_f.

Lemma III.26. *We have*

$$\Phi'_\pi(a) = \sum_{n\geq 0} \pi_n a^*_n \qquad \forall a \in l_f. \tag{III.11.8}$$

Proof. Let $\tilde{\Phi}$ be the function on l^+_f defined by

$$\tilde{\Phi}(a) := \sum_{n\geq 0} \pi_n a_n \qquad \forall a \in l^+_f.$$

In view of Lemma III.18, in order to prove that $\tilde{\Phi}$ agrees with Φ'_π it is enough to show that $\tilde{\Phi}$ satisfies the conditions (III.8)–(III.11) and agrees with Φ'_π on l^+_f.

Clearly, $\tilde{\Phi}$ satisfies (III.8)–(III.10). As, for all $a \in l_f$, we have

$$\sum_{n\geq 0} \pi_n a_n = \pi_0\sigma_0 + \sum_{n\geq 1} \pi_n(\sigma_{n+1}(a) - \sigma_n(a)) = \sum_{N\geq 1} (\pi_{N-1} - \pi_N)\sigma_N(a), \tag{III.11.9}$$

we see that $\tilde{\Phi}$ satisfies (III.11) as well.

It remains to prove that $\tilde{\Phi}$ agrees with Φ'_π on l_f^+. Let $a \in l_f^+$. Then by (III.13) we have

$$\Phi'_\pi(a) = \sup\left\{\sum_{n\geq 0} a_n b_n;\ b \in l_f^+,\ \Phi_\pi(b) = 1\right\}.$$

Notice that, for any $N \in \mathbb{N}$, the sequence $(\pi_0, \ldots, \pi_N, 0, 0, \ldots)$ belongs to l_f^+ and we can check that $\Phi_\pi(\pi_0, \ldots, \pi_N, 0, 0, \ldots) = 1$. Therefore, if N is large enough so that $a_n = 0$ for $n \geq N$, then

$$\Phi'_\pi(a) \geq \sum_{n\leq N} a_n\pi_n = \sum_{n\geq 0} a_n\pi_n = \tilde{\Phi}(a). \tag{III.11.10}$$

Let $b \in l_f^+$ be such that $\Phi_\pi(b) = 1$. Then $\sigma_N(b) \leq \sigma_N(\pi)$ for all $N \in \mathbb{N}$. Therefore, arguing as in (III.9), we get

$$\sum_{n\geq 0} a_n b_n = \sum_{N\geq 1} (a_N - a_{N-1})\sigma_N(b) \leq \sum_{N\geq 1} (a_N - a_{N-1})\sigma_N(\pi) = \sum_{n\geq 0} a_n\pi_n.$$

It then follows that $\Phi'_\pi(a) \leq \tilde{\Phi}(a)$. Combining this with (III.10) proves that $\tilde{\Phi}$ and Φ'_π agree on l_f^+. The proof is complete. □

It follows from Lemma III.26 that

$$\mathcal{I}_{\Phi'_\pi} = \left\{T \in \mathcal{K};\ \sum_{n\geq 0} \pi_n\mu_n(T) < \infty\right\} \quad \text{and} \quad \|T\|_{\Phi'_\pi} = \sum_{n\geq 0} \pi_n\mu_n(T) \quad \forall T \in \mathcal{K}.$$

Using (III.8) it is not hard to check that the symmetric norm Φ'_π satisfies (III.19). Therefore, from Proposition III.25 we get

Proposition III.35. *The Banach ideal $\mathcal{I}_{\Phi'_\pi}$ is separable and the finite-rank operators are dense in it, i.e., $\mathcal{I}^0_{\Phi'_\pi} = \mathcal{I}_{\Phi'_\pi}$.*

Using Proposition III.29, Lemma III.19 and the fact that $\mathcal{I}^0_{\Phi'_\pi} = \mathcal{I}_{\Phi'_\pi}$ we get

Proposition III.36. *The linear map (III.27) gives rise to isometric isomorphisms,*

$$\mathcal{I}_{\Phi'_\pi} \simeq (\mathcal{I}^0_{\Phi_\pi})' \qquad \text{and} \qquad \mathcal{I}_{\Phi_\pi} \simeq (\mathcal{I}_{\Phi'_\pi})'.$$

III.12. The Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$

Let $p \in (1, \infty)$. We denote by $\mathcal{L}^{(p,\infty)}$ the Banach ideal $\mathcal{I}_{\Phi_{(p,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(p,\infty)}(a) := \sup_{N\geq 1} \frac{\sigma_N(a)}{N^{1-\frac{1}{p}}} \qquad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(p,\infty)} = \left\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(N^{1-\frac{1}{p}})\right\},$$

and $\mathcal{L}^{(p,\infty)}$ is a Banach ideal for the norm,

$$\|T\|_{(p,\infty)} := \|T\|_{\Phi_{(p,\infty)}} = \sup_{N\geq 1} \frac{\sigma_N(T)}{N^{1-\frac{1}{p}}}.$$

Since $\sum_{n<N}(n+1)^{-\frac{1}{p}} \sim \frac{1}{1-\frac{1}{p}}N^{1-\frac{1}{p}}$ as $N \rightarrow \infty$, we see that $\Phi_{(p,\infty)}$ is equivalent to the symmetric norm $\Phi_{\pi^{(p)}}$ associated to the sequence $\pi^{(p)}$ in (III.2). Therefore, the Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{I}_{\Phi_{\pi^{(p)}}}$ have same underlying sets and their norms are equivalent. Since for $p > 1$ we have $\sigma_n(\pi^{(p)}) = \mathrm{O}(n\pi_n^{(p)})$, using Proposition III.33 and Proposition III.34 we obtain:

Proposition III.37. *Let $p \in (1, \infty)$. Then*

(1) *We have*

$$\mathcal{L}^{(p,\infty)} = \Big\{T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(n^{-\frac{1}{p}})\Big\}.$$

(2) *The Banach ideal $\mathcal{L}^{(p,\infty)}$ is not separable and the closure of finite-rank operators in $\mathcal{L}^{(p,\infty)}$ is*

$$\begin{aligned}\mathcal{L}_0^{(p,\infty)} &= \Big\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(N^{1-\frac{1}{p}})\Big\} \\ &= \Big\{T \in \mathcal{K};\ \mu_n(T) = \mathrm{o}(n^{-\frac{1}{p}})\Big\}.\end{aligned}$$

For $p = 1$ we let $\mathcal{L}^{(1,\infty)}$ be the Banach ideal $\mathcal{I}_{\Phi_{(1,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(1,\infty)}(a) := \sup_{N\geq 2} \frac{\sigma_N(a)}{\log N} \qquad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(1,\infty)} = \Big\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\log N)\Big\},$$

and the norm of $\mathcal{L}^{(p,\infty)}$ is

$$\|T\|_{(1,\infty)} := \|T\|_{\Phi_{(1,\infty)}} = \sup_{N\geq 2} \frac{\sigma_N(T)}{\log N}.$$

This Banach ideal is sometimes called the *Dixmier ideal*, since this is the natural domain of the Dixmier trace (cf. Chapter IV).

As in the case $p > 1$, the symmetric norms $\Phi_{(1,\infty)}$ and $\Phi_{\pi^{(1)}}$ are equivalent, so the Banach ideals $\mathcal{L}^{(1,\infty)}$ and $\mathcal{I}_{\Phi_{\pi^{(1)}}}$ have same underlying sets and their norms are equivalent. Therefore, using Proposition III.33 we get:

Proposition III.38. *The Banach ideal $\mathcal{L}^{(1,\infty)}$ is not separable and the closure of finite-rank operators in $\mathcal{L}^{(1,\infty)}$ is*

$$\mathcal{L}_0^{(1,\infty)} = \Big\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(\log N)\Big\}.$$

Remark III.24. Unlike in the case $p > 1$, we have a strict inclusion,

$$\mathcal{L}^{(1,\infty)} \supsetneq \Big\{T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(n^{-1})\Big\}.$$

Clearly, if $T \in \mathcal{K}$ is such that $\mu_n(T) = \mathrm{O}(n^{-1})$, then $\sigma_N(T) = \mathrm{O}(\log N)$, and hence T is contained in $\mathcal{L}^{(1,\infty)}$.

To show that the inclusion is strict we only have to exhibit a non-increasing sequence of positive numbers $(a_n)_{n\geq 0}$ such that $\sigma_N(a) = \mathrm{O}(\log N)$ and na_n is not bounded. An example of such a sequence is obtained as follows.

For any $k \in \mathbb{N}$ set $n_k = k^k$ and let $(a_n)_{n\geq 0}$ be the sequence defined by

$$a_0 = a_1 = 1 \qquad \text{and} \qquad a_n = \frac{1+\log k}{n_k} \quad \text{for } n_{k-1} < n \leq n_k.$$

As $n_k a_{n_k} = 1 + \log k \to \infty$ as $k \to \infty$, we see that a_n is not a $\mathrm{O}(n^{-1})$.

Furthermore, for $k \geq 3$, we have

$$\begin{aligned}\sum_{2\leq n\leq n_k} a_n = \sum_{2\leq j\leq k} \frac{1+\log j}{n_j}(n_j - n_{j-1}) &\leq \sum_{2\leq j\leq k}(1+\log j)\\ &\leq \int_1^k (1+\log x)dx = k\log k = \log n_k.\end{aligned}$$

Therefore, if $n_{k-1} \leq N \leq n_k$, then

$$\sum_{2\leq n<N} a_n \leq \sum_{2\leq n\leq n_k} a_n \leq \frac{\log n_k}{\log n_{k-1}} \log n_{k-1} \leq \frac{k\log k}{(k-1)\log(k-1)} \log N \leq C\log N,$$

where we have set $C := \sup_{k\geq 3} \frac{k\log k}{(k-1)\log(k-1)}$. This shows that $\sigma_N(a) = \mathrm{O}(\log N)$, concluding the remark.

Next, let $p \in (1,\infty]$ and denote by $\mathcal{L}^{(p,1)}$ the Banach ideal associated to the symmetric norm $\Phi_{(1,\infty)}$ on l_f defined by

$$\Phi_{(p,1)}(a) := \sum_{n\geq 0}(n+1)^{\frac{1}{p}-1}a_n \qquad \forall a \in l_f.$$

In other words,

$$\mathcal{L}^{(p,1)} = \Big\{T \in \mathcal{K};\ \sum_{n\geq 0}(n+1)^{\frac{1}{p}-1}\mu_n(T) < \infty\Big\},$$

and the norm of $\mathcal{L}^{(p,1)}$ is given by

$$\|T\|_{(p,1)} := \sum_{n\geq 0}(n+1)^{\frac{1}{p}-1}\mu_n(T).$$

When $p = \infty$ the Banach ideal $\mathcal{L}^{(\infty,1)}$ is called the *Macaev ideal*.

Let $p' \in [1,\infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Since Lemma III.26 shows that $\Phi_{(p,1)}$ is the dual norm $\Phi'_{\pi^{(p')}}$, we see that $\mathcal{L}^{(p,1)}$ is the Banach ideal $\mathcal{I}_{\Phi'_{\pi^{(p')}}}$. Therefore, Proposition III.35 and Proposition III.36 yield:

Proposition III.39. *Let $p \in (1,\infty]$ and $p' \in [1,\infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.*

(1) *The Banach ideal $\mathcal{L}^{(p,1)}$ is separable and agrees with the closure of finite-rank operators.*
(2) *The linear map (III.27) yields isomorphisms,*

$$\mathcal{L}^{(p,1)} \simeq (\mathcal{L}_0^{(p',\infty)})' \qquad \text{and} \qquad \mathcal{L}^{(p',\infty)} \simeq (\mathcal{L}^{(p,1)})'$$

Finally, the ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$ can be compared to the Schatten ideals.

Proposition III.40. *We have continuous inclusions,*

$$\mathcal{L}^p \subset \mathcal{L}_0^{(p,\infty)} \quad \textit{and} \quad \mathcal{L}^{(p,\infty)} \subset \mathcal{L}^q, \qquad 1 \leq p < q < \infty, \tag{III.12.1}$$

$$\mathcal{L}^q \subset \mathcal{L}^{(p,1)}, \qquad 1 \leq q < p \leq \infty, \tag{III.12.2}$$

$$\mathcal{L}^{(p,1)} \subset \mathcal{L}^p, \qquad 1 \leq p < \infty. \tag{III.12.3}$$

Proof. In view of Proposition III.39 it is enough to prove that we have continuous inclusions,

$$\mathcal{L}^p \subset \mathcal{L}_0^{(p,\infty)}, \qquad 1 \leq p < \infty,$$
$$\mathcal{L}^q \subset \mathcal{L}^{(p,1)}, \qquad 1 \leq q < p \leq \infty,$$

since the other continuous inclusions would follow by duality.

Let $p \in (1,\infty]$ and let $q \in [1,p)$. Let $T \in \mathcal{L}^q$. Let p' and q' be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The Hölder inequality gives

$$\|T\|_{(p,1)} = \sum_{n \geq 0} (n+1)^{-\frac{1}{p'}} \mu_n(T) \leq \left(\sum_{n \geq 0} (n+1)^{-\frac{q'}{p'}} \right)^{\frac{1}{q'}} \|T\|_q.$$

The fact that $q < p$ insures us that $q' > p'$, so the series $\sum_{n \geq 0} (n+1)^{-\frac{q'}{p'}}$ is convergent, so we see that $\mathcal{L}^q$ is contained in $\mathcal{L}^{(p,1)}$ and the inclusion is continuous.

Let $p \in [1,\infty)$ and let $T \in \mathcal{L}^p$. Using the Hölder inequality we see that, for any $N \in \mathbb{N}$,

$$\sigma_N(T) = \sum_{n<N} \mu_n(T) \leq \left(\sum_{n<N} 1^{p'} \right)^{\frac{1}{p'}} \left(\sum_{n<N} \mu_n(T)^p \right)^{\frac{1}{p}} \leq N^{1-\frac{1}{p}} \|T\|_p.$$

In view of the definition of the norm of $\mathcal{L}^{(p,\infty)}$ this implies that

$$\|T\|_{(p,\infty)} \leq \|T\|_p \qquad \forall T \in \mathcal{L}^p.$$

Thus $\mathcal{L}^p$ is contained in $\mathcal{L}^{(p,\infty)}$ and the inclusion is continuous. Since by Proposition III.31 the finite-rank operators are dense in $\mathcal{L}^p$, it follows that $\mathcal{L}^p$ is contained the closure of finite-rank operators in $\mathcal{L}^{(p,\infty)}$, that is, the ideal $\mathcal{L}_0^{(p,\infty)}$. Therefore, we actually have a continuous inclusion of $\mathcal{L}^p$ in $\mathcal{L}_0^{(p,\infty)}$. The proof is complete. □

Remark III.25. Let $T \in \mathcal{K}$ be such that

$$\mu_n(T) = (n+1)^{-\frac{1}{p}} \left(\log(n+2)\right)^{-\alpha} \qquad \forall n \in \mathbb{N}_0.$$

The following observations hold:

- If $\alpha = \frac{1}{p}$ and $p \in (1,\infty)$, then T is not in $\mathcal{L}^p$ and Proposition III.37 and Proposition III.38 insure us that T is in $\mathcal{L}_0^{(p,\infty)}$.
- If $\alpha = -1$ and $p \in [1,\infty)$, then T is contained in every ideal $\mathcal{L}^q$ with $q > p$. If $p > 1$, then using Proposition III.37 we see that T is not in $\mathcal{L}^{(p,\infty)}$. Likewise, when $p = 1$ the operator T is not in $\mathcal{L}^{(1,\infty)}$.
- If $\alpha = 1$ and $p \in (1,\infty)$, then T is in $\mathcal{L}^{(p,1)}$, but it is not in any ideal $\mathcal{L}^q$ with $q > p$.
- If $p \in (1,\infty]$ and $\alpha \in (p^{-1},1)$, then T is in $\mathcal{L}^p$, but not in $\mathcal{L}^{(p,1)}$.

This shows that all the inclusions in (III.1)–(III.3) are strict.

Bibliography

[Br] Brislawn, C.: *Traceable integral kernels on countably generated measure spaces.* Pacific J. Math. **150** (1991) 229–240.

[Co] Connes, A.: *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[Du] Duflo, M.: *Généralités sur les représentations induites. Représentations des Groupes de Lie Résolubles*, Monographies de la Soc. Math. de France, vol. 4, Dunod, Paris, 1972, pp. 93–119.

[Fo] Folland, G.: *Real analysis.* 2nd edition. John Wiley & Sons, Inc., New York, 1999.

[GK] Gohberg, I.C.; Krein, M.G.: *Introduction to the theory of linear nonselfadjoint operators.* Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[GVF] Gracia-Bondía, J.M.; Várilly, J.C.; Figueroa, H.: *Elements of noncommutative geometry.* Birkhäuser Boston, Boston, MA, 2001.

[Si] Simon, B.: *Trace ideals and their applications.* Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.

CHAPTER IV

Quantized Calculus

The quantized calculus of Connes [**Co**] aims to translate the main tools of the infinitesimal calculus into the operator theoretic language of quantum mechanics. It allows us to write down a dictionary between classical notions in the infinitesimal calculus and their quantum analogues.

IV.1. Noncommutative Infinitesimal Calculus

In the sequel we let $\mathcal{H}$ be a separable Hilbert space. In practice $\mathcal{H}$ comes from a spectral triple $(\mathcal{A}, cH, D)$ and we shall set

$$F := \mathrm{Sign}(D) = D|D|^{-1}.$$

The first few lines of this dictionary between classical notions in the infinitesimal calculus and their quantum analogues are the following:

Classical	Quantum
Complex variable	Operator on $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order $\alpha > 0$	Compact operator T such that $\mu_n(T) = \mathrm{O}(n^{-\alpha})$
Differential $df = \sum \frac{\partial}{\partial x^j} dx^j$	$da := [F, a]$.
Integral	Dixmier Trace $⨍$

The first two lines comes directly from quantum mechanics. In that formalism the observables are selfadjoint operators and the values that can be observed from an observable are given by its spectrum. In addition, as we saw in Chapter I, we have a holomorphic functional calculus for *any* bounded operator on $\mathcal{H}$, but there is a continuous functional calculus only for normal operators, including selfadjoint operators.

Intuitively, an infinitesimal can be thought as an object which is smaller than ϵ for any $\epsilon > 0$. For an operator T the condition $\|T\| < \epsilon$ for all $\epsilon > 0$ holds only when $T = 0$, but it can be relaxed into the following:

For any $\epsilon > 0$ there is a finite-dimensional subspace E such that $\|T|E^{\perp}\| < \epsilon$.

As shown by Proposition III.4 this latter condition is equivalent to T being compact.

By Proposition III.4 we also know that an operator T is compact if and only if its characteristic values $\mu_n(T)$ converge to 0 as $n \to \infty$. Thus the compactness of

T can be measured by the decay of its characteristic values. Thus, an infinitesimal operator of order α, $\alpha > 0$, is just a compact operator T such that

$$\mu_n(T) = \mathrm{O}(n^{-\alpha}).$$

If $\alpha > 1$ and we set $p := \alpha^{-1}$, then we see that the set of infinitesimal operators of order α agrees with the operator ideal $\mathcal{L}^{(p,\infty)}$ introduced in the previous chapter.

For $\alpha = 1$ every infinitesimal of order 1 is contained in the ideal $\mathcal{L}^{(1,\infty)}$, but there are operators in $\mathcal{L}^{(1,\infty)}$ that are not infinitesimal operators of order 1.

The following show that intuitive rules of the infinitesimal calculus are satisfied.

Lemma IV.1. *For $j = 1, 2$ let T_j be an infinitesimal operator of order α_j.*

(1) *$T_1 + T_2$ is an infinitesimal operator of order $\min(\alpha_1, \alpha_2)$.*
(2) *$T_1 T_2$ is an infinitesimal operator of order $\alpha_1\alpha_2$.*

Proof. Thanks to (III.1.7) we have

$$\mu_{m+n}(T_1 + T_2) \leq \mu_m(T_1) + \mu_n(T_2) \qquad \forall m, n \in \mathbb{N}_0. \tag{IV.1.1}$$

Let $n \in \mathbb{N}_0$. Since $n \geq 2[\frac{n}{2}]$, by (III.1.3) we have $\mu_n(T_1 + T_2) = \mu_{[\frac{n}{2}]+[\frac{n}{2}]}(T_1 + T_2)$, and hence using (IV.1.1) we get

$$\mu_n(T_1 + T_2) \leq \mu_{[\frac{n}{2}]}(T_1) + \mu_{[\frac{n}{2}]}(T_2). \tag{IV.1.2}$$

Set $\alpha = \min(\alpha_1, \alpha_2)$. Then, as $[\frac{n}{2}] \geq \frac{n-1}{2}$, we have

$$\mu_{[\frac{n}{2}]}(T_j) = \mathrm{O}\left(\left[\frac{n}{2}\right]^{-\alpha_j}\right) = \mathrm{O}(n^{-\alpha_j}) = \mathrm{O}(n^{-\alpha}). \tag{IV.1.3}$$

Combining this with (IV.1.2) then shows that $T_1 + T_2$ is an infinitesimal operator of order α.

Next, by (III.1.9) we have

$$\mu_{n+m}(T_1 T_2) \leq \mu_m(T_1)\mu_n(T_2) \qquad \forall m, n \in \mathbb{N}_0.$$

Therefore, for any $n \in \mathbb{N}_0$,

$$\mu_n(T_1 T_2) \leq \mu_{[\frac{n}{2}]+[\frac{n}{2}]}(T_1 T_2) \leq \mu_{[\frac{n}{2}]}(T_1)\mu_{[\frac{n}{2}]}(T_2).$$

Combining this with (IV.1.3) shows that $\mu_n(T_1 T_2) = \mathrm{O}(n^{-\alpha_1})\,\mathrm{O}(n^{-\alpha_2}) = \mathrm{O}(n^{-(\alpha_1+\alpha_2)})$, that is, $T_1 T_2$ is an infinitesimal operator of order $\alpha_1\alpha_2$. The lemma is proved. $\square$

The differential $da := [F, a]$ is a derivation, and hence it satisfies Leibniz's Rule,

$$d(ab) = (da)b + adb, \qquad a, b \in \mathcal{A}.$$

In practice, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable for some $p \geq 1$, that is,

$$\mu_n(D^{-1}) = \mathrm{O}(n^{-\frac{1}{p}}).$$

This means that D^{-1} is an infinitesimal operator of order $\frac{1}{p}$.

Lemma IV.2 (cf. Chapter 11). *If $(\mathcal{A}, \mathcal{H}, D)$ is p-summable, then*

$$\mu_n([F, a]) = \mathrm{O}(n^{-\frac{1}{p}}) \qquad \forall a \in \mathcal{A}.$$

In other words, if the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable, then all the differentials $da = [F, a]$, $a \in \mathcal{A}$, are infinitesimal operators of order $\frac{1}{p}$.

If $x^1, \cdots, x^p$ form a system of local coordinates on a manifold M of dimension p, then $dx^1 \wedge \cdots \wedge dx^p$ can be thought as an "infinitesimal" of order 1 and the differentials $dx^1, \ldots, dx^p$ as "infinitesimals" of order $\frac{1}{p}$. We see that, similarly, if a^0, ..., a^p are in $\mathcal{A}$, then da^1, ..., da^p are infinitesimal operators of order $\frac{1}{p}$ and $a^0 da^1 \cdots da^p$ is an infinitesimal operator of order 1.

Next, the classical integral is a linear functional which the following properties:

(i) It is defined on infinitesimals of order 1.
(ii) It vanishes on infinitesimals of order > 1, i.e., we can integrate by neglecting higher-order infinitesimals (e.g., by approximating an integral by Riemann sums).
(iii) It is positive, i.e., the integral of a non-negative function is a non-negative number.
(iv) It vanishes on total differentials.

In the setting of quantized calculus we thus seek for a linear functional satisfying noncommutative analogues of the above conditions. The first three conditions can be easily translated into:

(i') Its domain contains the infinitesimal operators of order 1.
(ii') It vanishes on infinitesimal operators of order > 1.
(iii') It is positive, i.e., it takes on non-negative real values on the operators in its domain that are positive.

At least in the p-summable case, the condition (iv) corresponds to the vanishing of the quantum integral on operators of the form

$$d(a^0 da^1 \cdots da^p) = [F, a^0 da^1 \cdots da^p] \qquad a^j \in \mathcal{A}.$$

In general, we require the noncommutative integral to vanish on commutators,

$$[A, T],$$

where A ranges over $\mathcal{L}(\mathcal{H})$ and T ranges over the domain of the quantum integral. Thus we require

(iv') The quantum integral is a trace.

One candidate that comes to mind is the operator trace $T \rightarrow \operatorname{Trace}(T)$. This is a positive trace, but it satisfies none of the conditions (i')–(ii'). More precisely:

- The domain of the operator trace is the ideal $\mathcal{L}^1$ of trace-class operators, but an infinitesimal operator of order 1 need not be trace-class, e.g., if $\mu_n(T) = (n+1)^{-1}$, then $\sum \mu_n(T) = \infty$, and hence T is not trace-class.
- The operator trace does not vanish on all infinitesimal operators of order > 1, e.g., it does not vanish on (non-zero) finite-rank projections.

The solution for finding a positive linear trace satisfying (i')–(ii') is actually provided by the Dixmier trace. This trace was constructed by Dixmier [**Di**] as an example of a non-normal trace on $\mathcal{L}(\mathcal{H})$.

The rest of the chapter is devoted to presenting the construction of the Dixmier trace. The exposition follows closely [**CM**, Appendix A] (see also [**GVF**, Section 7.5]).

IV.2. The Dixmier Ideal $\mathcal{L}^{(1,\infty)}$

As we shall see in the next section, the domain of the Dixmier trace is the Dixmier ideal $\mathcal{L}^{(1,\infty)}$ introduced in the previous chapter. It is defined as follows.

For $T \in \mathcal{K}$ set

$$\|T\|_{(1,\infty)} := \sup_{N \geq 2} \frac{\sigma_N(T)}{\log N}.$$

We then define

$$\mathcal{L}^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \|T\|_{(1,\infty)} < \infty \Big\}.$$

Equivalently,

$$\mathcal{L}^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\log N) \Big\}.$$

Proposition IV.1 (See Chapter III). *The following hold.*

(1) $\mathcal{L}^{(1,\infty)}$ *is a two-sided ideal and* $\|.\|_{(1,\infty)}$ *is a norm on* $\mathcal{L}^{(1,\infty)}$ *for which* $\mathcal{L}^{(1,\infty)}$ *is a Banach ideal. In particular,*

$$\|ATB\|_{(1,\infty)} \leq \|A\|\|T\|_{(1,\infty)}\|B\| \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{IV.2.1}$$

(2) *The Banach ideal* $\mathcal{L}^{(1,\infty)}$ *is not separable.*

(3) *Let* $\mathcal{L}_0^{(1,\infty)}$ *be the closure of finite-rank operators in* $\mathcal{L}^{(1,\infty)}$*. Then*

$$\mathcal{L}_0^{(1,\infty)} = \Big\{T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(\log N)\Big\}. \tag{IV.2.2}$$

(4) *There are continuous inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}_0^{(1,\infty)} \qquad \text{and} \qquad \mathcal{L}^{(1,\infty)} \subset \mathcal{K}.$$

(5) *There is a strict inclusion,*

$$\mathcal{L}^{(1,\infty)} \supsetneq \Big\{T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(\frac{1}{n})\Big\}. \tag{IV.2.3}$$

Remark IV.1. It can be shown that a Banach ideal is separable if and only if the finite-rank operators are dense in it (see Chapter III). Therefore (IV.2.2) implies that $\mathcal{L}^{(1,\infty)}$ is not separable. Notice that $\mathcal{L}_0^{(1,\infty)}$ is a Banach ideal with respect to the norm $\|.\|_{(1,\infty)}$, since this is the closure in $\mathcal{L}^{(1,\infty)}$ of the ideal of finite-rank operators.

Remark IV.2. The continuous inclusions (IV.2.3) hold for any non-trivial Banach ideal (see Chapter III). In the case of $\mathcal{L}^{(1,\infty)}$, the inclusion of $\mathcal{L}^1$ in $\mathcal{L}_0^{(1,\infty)}$ follows from the fact that if $T \in \mathcal{L}^1$, then $\sigma_N(T) = \mathrm{O}(1) = \mathrm{o}(\log N)$. The continuity of the inclusions (IV.2.3) can also be deduced from the fact that, for any $T \in \mathcal{K}$, we have

$$\|T\| = \mu_0(T) \leq \sigma_N(T) \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1 \qquad \forall N \in \mathbb{N},$$

which implies that

$$(\log 2)^{-1}\|T\| \leq \|T\|_{(1,\infty)} \leq (\log 2)^{-1}\|T\|_1 \qquad \forall T \in \mathcal{K}.$$

The Banach ideal $\mathcal{L}^{(1,\infty)}$ is in duality with the Macaev ideal $\mathcal{L}^{(\infty,1)}$. The latter can be defined as follows.

For $T \in \mathcal{K}$ we set

$$\|T\|_{(\infty,1)} := \sum_{n\geq 0}(n+1)^{-1}\mu_n(T).$$

We then define

$$\mathcal{L}^{(\infty,1)} = \Big\{T \in \mathcal{K};\ \|T\|_{(\infty,1)} < \infty\Big\}.$$

Proposition IV.2 (See Chapter III). *The following hold.*

(1) $\mathcal{L}^{(\infty,1)}$ *is a two-sided ideal and* $\|.\|_{(\infty,1)}$ *is a norm on* $\mathcal{L}^{(\infty,1)}$ *for which* $\mathcal{L}^{(\infty,1)}$ *is a Banach ideal.*

(2) *The Banach ideal* $\mathcal{L}^{(\infty,1)}$ *is separable and the finite-rank operators are dense in it.*

(3) *There are continuous inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}^{(\infty,1)} \subset \mathcal{K}.$$

(4) *There are isomorphisms,*

$$\mathcal{L}^{(1,\infty)} \simeq (\mathcal{L}^{(\infty,1)})' \qquad \text{and} \qquad \mathcal{L}^{(\infty,1)} \simeq (\mathcal{L}_0^{(1,\infty)})'. \tag{IV.2.4}$$

Remark IV.3. As explained in Chapter III, the duality isomorphisms (IV.2.4) are inherited from the isometric isomorphism,

$$\mathcal{L}(\mathcal{H}) \ni S \longrightarrow (S,.) \in (\mathcal{L}^1)', \qquad (S,T) = \operatorname{Trace}(ST) \quad \forall T \in \mathcal{L}^1.$$

More precisely, if $S \in \mathcal{L}^{(\infty,1)}$ and $T \in \mathcal{L}^{(1,\infty)}$, then ST is trace-class and $(S,.)$ uniquely extends to a continuous linear form on $\mathcal{L}_0^{(1,\infty)}$. This yields the isomorphism from $\mathcal{L}^{(\infty,1)}$ onto $(\mathcal{L}_0^{(1,\infty)})'$. Similarly, if $S \in \mathcal{L}^{(1,\infty)}$, then $(S,.)$ uniquely extends to a continuous linear form on $\mathcal{L}^{(\infty,1)}$, which allows us to get an isomorphism from $\mathcal{L}^{(1,\infty)}$ onto $(\mathcal{L}^{(\infty,1)})'$. Notice also that these isomorphisms become isometries if we replace the norm $\|.\|_{(1,\infty)}$ by the equivalent norm,

$$\|T\|'_{(1,\infty)} := \sup_{N\geq 1} \frac{\sigma_N(T)}{\sum_{n<N}(n+1)^{-1}}.$$

In the sequel we let $\mathcal{H}'$ be a (separable) Hilbert space.

Lemma IV.3. *Let Φ be a continuous $*$-homomorphism from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}')$ such that*

$$\mu_n(\Phi(T)) = \mu_n(T) \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall n \in \mathbb{N}_0. \tag{IV.2.5}$$

Then Φ induces an isometric linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$.

Proof. Using Proposition III.4-(iii) and (IV.2.5) we see that Φ maps $\mathcal{K}(\mathcal{H})$ to $\mathcal{K}(\mathcal{H}')$. Moreover, if $T \in \mathcal{K}$, then (IV.2.5) implies that $\sigma_N(\Phi(T)) = \sigma_N(T)$ for all $N \in \mathbb{N}$, and hence $\|\Phi(T)\|_{(1,\infty)} = \|T\|_{(1,\infty)}$. Thus Φ gives rise to an isometric linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. The lemma is proved. ☐

Let $S : \mathcal{H}' \to \mathcal{H}$ be a continuous linear isomorphism from $\mathcal{H}'$ onto $\mathcal{H}$. We denote by γ_S the conjugation by S, i.e., the map

$$\mathcal{L}(\mathcal{H}) \ni T \longrightarrow S^{-1}TS \in \mathcal{L}(\mathcal{H}').$$

This a continuous isomorphism from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H}')$.

Proposition IV.3. *The conjugation by S gives rise to a continuous isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, and hence*

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}') = S^{-1}\mathcal{L}^{(1,\infty)}(\mathcal{H})S.$$

Furthermore, if S is unitary, then this isomorphism is isometric.

Proof. Let $S \in \mathcal{L}(\mathcal{H})$ be invertible. Since $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ both γ_S and its inverse $\gamma_S^{-1} = \gamma_{S^{-1}}$ maps $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to itself, and hence γ_S induces a linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to itself. Furthermore, this linear map is continuous, for by (IV.2.1) we have

$$\|S^{-1}TS\|_{(1,\infty)} \leq \|S\|\|S^{-1}\|\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Let $U \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ be unitary. Then by Remark III.2 we have

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall T \in \mathcal{L}(\mathcal{H}),$$

that is, γ_U satisfies (IV.2.5). Thus, it follows from Lemma IV.3 that γ_U induces a linear isometry from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Similarly, its inverse $\gamma_U^{-1} = \gamma_{U^*}$ induces a linear isometry from $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, so γ_U induces an isometric linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$.

Now, let $S \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ be a general isomorphism. Set $|S| = (S^*S)^{\frac{1}{2}}$ and $U = S|S|^{-1}$. Then $|S|$ is an invertible element of $\mathcal{L}(\mathcal{H}')$ and U is an unitary element of $\mathcal{L}(\mathcal{H}', \mathcal{H})$, for it is invertible and, as $|S|^{-1}S^*S|S|^{-1} = |S|^{-1}|S|^2|S|^{-1}$, for any $\xi \in \mathcal{H}'$, we have

$$\|U\xi\|^2_{\mathcal{H}} = \langle S|S|^{-1}\xi, S|S|^{-1}\xi\rangle_{\mathcal{H}} = \langle \xi, |S|^{-1}S^*S|S|^{-1}\xi\rangle_{\mathcal{H}'} = \langle \xi, \xi\rangle_{\mathcal{H}} = \|\xi\|^2_{\mathcal{H}'}.$$

Thus, by the first two parts of the proof $\gamma_{|S|}$ induces a continuous linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$ onto itself and γ_U induces an isometric linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Since $S = U|S|$, and hence $\gamma_S = \gamma_{|S|} \circ \gamma_U$, it follows that γ_S a continuous linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, proving the proposition. $\square$

In particular, if we let $\mathcal{H}'$ be the Hilbert space with same underlying vector space as $\mathcal{H}$ and equipped with an equivalent inner product and we let S be the identity map, then we obtain:

Corollary IV.1. *Neither $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ nor its topology depend on the choice of the inner product of $\mathcal{H}$.*

IV.3. The Dixmier Trace

In this section, we shall construct the Dixmier trace as a trace on the Banach ideal $\mathcal{L}^{(1,\infty)}$. It will occur from the analysis of the logarithmic divergency of the partial traces,

$$\sigma_N(T) = \sum_{n<N} \mu_n(T), \qquad T \in \mathcal{K}, \quad N \in \mathbb{N}.$$

The first step is to extend the definition of $\sigma_N(T)$ to non-integer values of N. To this end recall that by Proposition III.21 we have

$$\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T) \qquad \forall S, T \in \mathcal{K}. \tag{IV.3.1}$$

Lemma IV.4. *Let $N \in \mathbb{N}$. Then, for all $T \in \mathcal{K}$,*

$$\sigma_N(T) = \inf\{\|x\|_1 + N\|y\|;\ (x,y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x+y=T\}. \tag{IV.3.2}$$

Proof. Let $T \in \mathcal{K}$. Let $(x,y) \in \mathcal{L}^1 \times \mathcal{K}$ be such that $x+y=T$. Using (IV.3.1) we get

$$\sigma_N(T) = \sigma_N(x+y) \leq \sigma_N(x) + \sigma_N(y).$$

Notice that by (III.9.5) we have $\sigma_N(y) \leq N\|y\|$. Moreover,

$$\sigma_N(x) = \sum_{n<N} \mu_n(x) \leq \sum_{n\geq 0} \mu_n(x) = \|x\|_1.$$

Therefore, we have

$$\sigma_N(T) \leq \|x\|_1 + N\|y\|.$$

It then follows that

$$\sigma_N(T) \leq \inf\{\|x\|_1 + N\|y\|;\ (x,y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x+y=T\}. \tag{IV.3.3}$$

Let $T = U|T|$ be the polar decomposition of T, and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Denote by Π_N the orthogonal projection onto the span of $\xi_0, \ldots, \xi_{N-1}$. Define

$$x_N := (|T| - \mu_N(T))\Pi_N \qquad \text{and} \qquad y_N := \mu_N(T)\Pi_N + |T|(1-\Pi_N). \tag{IV.3.4}$$

Notice that $x_N + y_N = |T|$. Thus,

$$T = U|T| = Ux_N + Uy_N. \tag{IV.3.5}$$

Notice also that, as Ux_N has finite-rank, this is a trace-class operator. In addition, as $\Pi_N = \sum_{n<N} \xi_n \otimes \xi_n^*$ and, by (III.1.15), $|T| = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*$, we have

$$x_N = \sum_{n<N} (\mu_n(T) - \mu_N(T))\xi_n \otimes \xi_n^*, \tag{IV.3.6}$$

$$y_N = \sum_{n<N} \mu_N(T)\xi_n \otimes \xi_n^* + \sum_{n\geq N} \mu_n(T)\xi_n \otimes \xi_n^*. \tag{IV.3.7}$$

It follows from (IV.3.6) and the min-max principle that $\mu_n(x_N)$ is equal to $\mu_n(T) - \mu_N(T)$ for $n < N$ and is zero for $n \geq N$. Thus,

$$\|x_N\|_1 = \sum_{n\geq 0} \mu_n(x_N) = \sum_{n<N} (\mu_n(T) - \mu_N(T)) = \sigma_N(T) - N\mu_N(T).$$

Since by Proposition II.3 $\|U\| \leq 1$, combining this with (III.2.5) gives

$$\|Ux_N\|_1 \leq \|U\|\|x_N\|_1 \leq \sigma_N(T) - N\mu_N(T). \tag{IV.3.8}$$

As for y_N, it follows from (IV.3.7) that y_N is a positive operator whose greatest eigenvalue is $\mu_N(T)$, and so using the min-max principle we get

$$\|y_N\| = \mu_0(y_N) = \mu_N(T),$$

and hence

$$\|Uy_N\| \leq \|U\|\|y_N\| \leq \mu_N(T). \tag{IV.3.9}$$

Combining this with (IV.3.8) gives

$$\|Ux_N\|_1 + N\|Uy_N\| \leq \sigma_N(T) - N\mu_N(T) + N\mu_N(T) = \sigma_N(T).$$

In view of (IV.3.5) it follows that

$$\sigma_N(T) \geq \inf\{\|x\|_1 + N\|y\|;\ (x, y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x + y = T\}.$$

Combining this with (IV.3.3) then proves the lemma. □

The previous lemma allow us to extend the definition of σ_N to non-integer values of N.

Definition IV.4. *Let $T \in \mathcal{K}$. Then, for any $\lambda \geq 0$, we define*

$$\sigma_\lambda(T) = \inf\{\|x\|_1 + N\|y\|;\ (x, y) \in \mathcal{L}^1 \times \mathcal{K} \textit{ and } x + y = T\}.$$

Lemma IV.4 shows that, when λ is an integer, the above definition of $\sigma_\lambda(T)$ agrees with that given in (III.9.2).

Lemma IV.5. *Let $T \in \mathcal{K}$. Then*

(i) *The function $\lambda \to \sigma_\lambda(T)$ is concave.*

(ii) *For any $\lambda \geq 0$, we have*

$$\sigma_\lambda(T) = \sigma_N(T) + \alpha\mu_N(T), \tag{IV.3.10}$$

$$= (1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T), \tag{IV.3.11}$$

where we have set $N = [\lambda]$ and $\alpha = \lambda - [\lambda]$.

Proof. Let $\lambda, \mu \in [0, \infty)$ and let $\alpha \in [0, 1]$. For any $(x, y) \in \mathcal{L}^1 \times \mathcal{K}$ be such that $x + y = T$ we have

$$\begin{aligned}\|x\|_1 + (\alpha\lambda + (1-\alpha)\mu)\|y\| &= \alpha(\|x\|_1 + \lambda\|y\|) + (1-\alpha)(\|x\|_1 + \mu\|y\|)\\ &\geq \alpha\sigma_\lambda(T) + (1-\alpha)\sigma_\mu(T).\end{aligned}$$

Thus,

$$\sigma_{\alpha\lambda+(1-\alpha)\mu}(T) \geq \alpha\sigma_\lambda(T) + (1-\alpha)\sigma_\mu(T),$$

which shows that the function $\lambda \to \sigma_\lambda(T)$ is concave.

Let $\lambda \in [0, \infty)$ and set $N = [\lambda]$ and $\alpha = \lambda - N$. As $\sigma_{N+1}(T) = \sigma_N(T) + \mu_N(T)$, we have

$$\begin{aligned}(1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T) &= (1-\alpha)\sigma_N(T) + \alpha(\sigma_N(T) + \mu_N(T))\\ &= \sigma_N(T) + \alpha\mu_N(T).\end{aligned} \tag{IV.3.12}$$

Notice also that $\lambda = (1-\alpha)N + \alpha(N+1)$. As the function $\lambda \to \sigma_\lambda(T)$ is concave, it follows that

$$\sigma_\lambda(T) \geq (1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T). \tag{IV.3.13}$$

Let $T = U|T|$ be the polar decomposition of T and let x_N and y_N be as in (IV.3.4), so that $T = Ux_N + Uy_N$ and, as in (IV.3.8) and (IV.3.9), we have

$$\|Ux_N\|_1 \leq \sigma_N(T) - N\mu_N(T) \quad \text{and} \quad \|Uy_N\| \leq \mu_N(T).$$

Then

$$\sigma_\lambda(T) \leq \|Ux_N\|_1 + \lambda\|y_N\| \leq \sigma_N(T) + (\lambda - N)\mu_N(T).$$

Combining this with (IV.3.12) and (IV.3.13) proves (IV.3.10) and (IV.3.11). The proof is complete. □

Remark IV.5. The equality (IV.3.10) can be rewritten as

$$\sigma_\lambda(T) = \int_0^\lambda \mu_{[u]}(T)du.$$

Thus, when T is positive, $\sigma_\lambda(T)$ can be seen as the cut-off by λ of the trace,

$$\operatorname{Trace} T = \int_0^\infty \mu_{[u]}(T)du.$$

Remark IV.6. It follows from (IV.3.11) that the function $\lambda \to \sigma_\lambda(T)$ is affine between $\sigma_N(T)$ and $\sigma_{N+1}(T)$, so this function agrees with the affine interpolation of the $\sigma_N(T)$'s.

Remark IV.7. Since the σ_N's are norms on $\mathcal{K}$, it follows from (IV.3.11) that the σ_λ's too are norms. Thus, for any $\lambda \geq 0$, we have

$$\sigma_\lambda(cT) = |c|\sigma_\lambda(T) \qquad \forall T \in \mathcal{K}\ \forall c \in \mathbb{C}, \tag{IV.3.14}$$

$$\sigma_\lambda(S+T) \leq \sigma_\lambda(S) + \sigma_\lambda(T) \qquad \forall S, T \in \mathcal{K}. \tag{IV.3.15}$$

Lemma IV.6. *Let T_1 and T_2 be positive compact operators. Then*

$$\sigma_{\lambda_1+\lambda_2}(T_1+T_2) \geq \sigma_{\lambda_1}(T_1) + \sigma_{\lambda_2}(T_2) \qquad \forall \lambda_j \geq 0. \tag{IV.3.16}$$

Proof. For $j = 1, 2$ let $N_j \in \mathbb{N}$ and let us show that

$$\sigma_{N_1+N_2}(T_1+T_2) \geq \sigma_{N_1}(T_1) + \sigma_{N_2}(T_2). \tag{IV.3.17}$$

For $j = 1, 2$ let E_j be a subspace of $\mathcal{H}$ of dimension N_j, and let E be a subspace of dimension $N_1 + N_1$ containing E_1 and E_2. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $\xi_0, \ldots, \xi_{N_1-1}$ span E_1 and $\xi_0, \ldots, \xi_{N-1}$ span E. Then

$$\operatorname{Trace}(T_1\Pi_{E_1}) = \sum_{n\geq 0}\langle \xi_n, T_1\Pi_{E_1}\xi_n\rangle = \sum_{n<N_1}\langle \xi_n, T_1\xi_n\rangle.$$

Similarly, we have

$$\operatorname{Trace}(T_1\Pi_E) = \sum_{n<N}\langle \xi_n, T_1\xi_n\rangle.$$

As T_1 is positive $\langle \xi_n, T_1\xi_n\rangle \geq 0$ for all $n \in \mathbb{N}_0$, and hence

$$\operatorname{Trace}(T_1\Pi_{E_1}) \leq \operatorname{Trace}(T_1\Pi_E).$$

Similarly $\operatorname{Trace}(T_2\Pi_{E_2}) \leq \operatorname{Trace}(T_2\Pi_E)$, and hence using (III.9.7) we get

$$\operatorname{Trace}(T_1\Pi_{E_1}) + \operatorname{Trace}(T_2\Pi_{E_2}) \leq \operatorname{Trace}\left((T_1+T_2)\Pi_E)\right) \leq \sigma_{N_1+N_2}(T_1+T_2).$$

Thanks to (III.9.7) taking the supremum of $\operatorname{Trace}(T_1\Pi_{E_1}) + \operatorname{Trace}(T_2\Pi_{E_2})$ over all subspaces E_1 of dimension N_1 and all subspaces E_2 of dimension N_2 gives (IV.3.17).

Now, for $j=1,2$ let λ_j be a non-negative real number and set $N_j=[\lambda_j]$ and $\alpha_j=\lambda_j-N_j$. In addition, define $\lambda=\lambda_1+\lambda_2$ and set $N=[\lambda]$ and $\alpha=\lambda-N$. Notice that either $N=N_1+N_2$ or $N=N_1+N_2+1$.

Assume that $N=N_1+N_2$. Then $\alpha=\alpha_1+\alpha_2$, and hence by (IV.3.11) we have

$$\begin{aligned}\sigma_\lambda(T_1+T_2)&=(1-\alpha_1-\alpha_2)\sigma_N(T_1+T_2)+(\alpha_1+\alpha_2)\sigma_{N+1}(T_1+T_2)\\&=(1-\alpha_1-\alpha_2)\sigma_{N_1+N_2}(T_1+T_2)+(\alpha_1+\alpha_2)\sigma_{N_1+N_2+1}(T_1+T_2).\end{aligned}\tag{IV.3.18}$$

By (IV.3.17) we have

$$\sigma_{N_1+N_2}(T_1+T_2)\geq\sigma_{N_1}(T_1)+\sigma_{N_2}(T_2),\tag{IV.3.19}$$

$$\sigma_{N_1+N_2+1}(T_1+T_2)\geq\sigma_{N_1+1}(T_1)+\sigma_{N_2}(T_2),\tag{IV.3.20}$$

$$\sigma_{N_1+N_2+1}(T_1+T_2)\geq\sigma_{N_1}(T_1)+\sigma_{N_2+1}(T_2).\tag{IV.3.21}$$

Combining this with (IV.3.11) and (IV.3.18) we get

$$\begin{aligned}\sigma_\lambda(T_1+T_2)\geq{}&(1-\alpha_1-\alpha_2)(\sigma_{N_1}(T_1)+\sigma_{N_2}(T_2))\\&+\alpha_1(\sigma_{N_1+1}(T_1)+\sigma_{N_2}(T_2))+\alpha_2(\sigma_{N_1}(T_1)+\sigma_{N_2+1}(T_2)),\\\geq{}&(1-\alpha_1)\sigma_{N_1}(T_1)+\alpha_1\sigma_{N_1+1}(T_1)+(1-\alpha_2)\sigma_{N_2}(T_2)+\alpha_2\sigma_{N_2+1}(T_2)\\\geq{}&\sigma_{\lambda_1}(T_1)+\sigma_{\lambda_2}(T_2).\end{aligned}\tag{IV.3.22}$$

Suppose now that $N=N_1+N_2+1$. Then $\alpha=\alpha_1+\alpha_2-1$, and hence (IV.3.11) gives

$$\begin{aligned}&\sigma_\lambda(T_1+T_2)=(2-\alpha_1-\alpha_2)\sigma_{N+1}(T_1+T_2)+(\alpha_1+\alpha_2-1)\sigma_{N+2}(T_1+T_2)\\&=[(1-\alpha_1)+(1-\alpha_2)]\sigma_{N_1+N_2+1}(T_1+T_2)+(\alpha_1+\alpha_2-1)\sigma_{N_1+1+N_2+1}(T_1+T_2).\end{aligned}$$

By (IV.3.17) we have

$$\sigma_{N_1+1+N_2+1}(T_1+T_2)\geq\sigma_{N_1+1}(T_1)+\sigma_{N_2+1}(T_2),$$

Combining this with (IV.3.11) and (IV.3.19)–(IV.3.21) we get

$$\begin{aligned}\sigma_\lambda(T_1+T_2)\geq{}&(1-\alpha_1)(\sigma_{N_1}(T_1)+\sigma_{N_2+1}(T_2))+(1-\alpha_2)(\sigma_{N_1+1}(T_1)+\sigma_{N_2}(T_2))\\&+(\alpha_1+\alpha_2-1)(\sigma_{N_1+1}(T_1)+\sigma_{N_2+1}(T_2)),\\\geq{}&(1-\alpha_1)\sigma_{N_1}(T_1)+\alpha_1\sigma_{N_1+1}(T_1)+(1-\alpha_2)\sigma_{N_2}(T_2)+\alpha_2\sigma_{N_2+1}(T_2),\\\geq{}&\sigma_{\lambda_1}(T_1)+\sigma_{\lambda_2}(T_2).\end{aligned}$$

The proof is complete. □

In the sequel we denote by $\mathcal{L}^{(1,\infty)}_+$ the cone of positive operators of $\mathcal{L}^{(1,\infty)}$.

Let $T\in\mathcal{L}^{(1,\infty)}_+$. For $\lambda\geq e$ we define

$$\tau_\lambda(T):=\frac{1}{\log\lambda}\int_e^\lambda\frac{\sigma_u(T)}{\log u}\frac{du}{u}.\tag{IV.3.23}$$

In other words, the function $\lambda\rightarrow\tau_\lambda(T)$ is the Cesāro mean of $\frac{\sigma_\lambda(T)}{\log\lambda}$ with respect to the Haar measure $\frac{du}{u}$ of the multiplicative group $\mathbb{R}^*_+$.

Lemma IV.7. *We have*

$$\sigma_\lambda(T)\leq2\|T\|_{(1,\infty)}\log\lambda\qquad\forall\lambda\geq2.\tag{IV.3.24}$$

$$0\leq\tau_\lambda(T)\leq2\|T\|_{(1,\infty)}\qquad\forall\lambda\geq e.\tag{IV.3.25}$$

Proof. Let $\lambda\in[2,\infty)$ and set $N=[\lambda]$. Then

$$\frac{\sigma_\lambda(T)}{\log\lambda}\leq\frac{\sigma_{N+1}}{\log N}\leq\frac{\log(N+1)}{\log N}.\frac{\sigma_{N+1}(T)}{\log(N+1)}\leq2\|T\|_{(1,\infty)}$$

where we have used the fact that $\sup_{u\geq 2}\frac{\log(u+1)}{\log u}=\frac{\log 2}{\log 3}\leq 2$. It follows from this, that for all $\lambda\geq e$, we have

$$\tau_\lambda(T)=\frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u}\leq \frac{1}{\log\lambda}\int_e^\lambda 2\|T\|_{(1,\infty)}\frac{du}{u}\leq 2\|T\|_{(1,\infty)}.$$

The lemma is proved. $\square$

In particular, this lemma shows that the function $\lambda\rightarrow\tau_\lambda(T)$ is contained in $C_b[e,\infty)$, the C^*-algebra of bounded continuous functions on the interval $[e,\infty)$.

The interest of considering the above Cesāro mean stems from the fact that, while $\frac{\sigma_\lambda(T)}{\log\lambda}$ is not additive with respect to T, its Cesāro mean is asymptotically additive as $\lambda\rightarrow\infty$. Namely, we have:

Lemma IV.8. *Let T_1 and T_2 be in $\mathcal{L}_+^{(1,\infty)}$. Then, for all $\lambda\geq e$,*

$$|\tau_\lambda(T_1+T_2)-\tau_\lambda(T_1)-\tau_\lambda(T_2)|\leq 2\|T_1+T_2\|_{(1,\infty)}\frac{(\log(\log\lambda)+2)}{\log\lambda}.$$

Proof. First, it follows from (IV.3.15) that

$$\tau_\lambda(T_1+T_2)\leq\tau_\lambda(T_1)+\tau_\lambda(T_2)\qquad\forall\lambda\geq e. \tag{IV.3.26}$$

It remains to find an upper bound for $\tau_\lambda(T_1)+\tau_\lambda(T_2)-\tau_\lambda(T_1+T_2)$. By Lemma IV.6 we have

$$\sigma_u(T_1)+\sigma_u(T_2)\leq\sigma_{2u}(T_1+T_2)\qquad\forall u\geq e.$$

Thus,

$$\tau_\lambda(T_1)+\tau_\lambda(T_2)\leq\frac{1}{\log\lambda}\int_e^\lambda\frac{\sigma_{2u}(T_1+T_2)}{\log u}\frac{du}{u}=\frac{1}{\log\lambda}\int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log(\frac{u}{2})}\frac{du}{u}.$$

Since $\tau_\lambda(T_1+T_2)=\frac{1}{\log\lambda}\int_e^\lambda\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}$ we deduce that

$$\begin{aligned}(\log\lambda)\{\tau_\lambda(T_1)+\tau_\lambda(T_2)-\tau_\lambda(T_1+T_2)\}\\ \leq\int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log(\frac{u}{2})}\frac{du}{u}-\int_e^\lambda\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}=\delta+\delta',\end{aligned}\tag{IV.3.27}$$

where we have let

$$\delta=\int_{2e}^{2\lambda}\sigma_u(T_1+T_2)\left(\frac{1}{\log(\frac{u}{2})}-\frac{1}{\log u}\right)\frac{du}{u},$$
$$\delta'=\int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}-\int_e^\lambda\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}.$$

Notice that $\frac{1}{\log(\frac{u}{2})}-\frac{1}{\log u}=\frac{\log 2}{\log u\log\frac{u}{2}}$. Since $\sigma_u(T_1+T_2)\leq 2\|T_1+T_2\|_{(1,\infty)}\log u$, we then see that

$$\delta\leq 2\|T_1+T_2\|_{(1,\infty)}\log 2\int_{2e}^{2\lambda}\frac{1}{\log\frac{u}{2}}\frac{du}{u}=2\|T_1+T_2\|_{(1,\infty)}(\log 2)\log(\log\lambda). \tag{IV.3.28}$$

In addition, we have

$$\begin{aligned}\delta'=\int_{2e}^{e}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}+\int_e^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}+\int_\lambda^e\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}\\ \leq\int_\lambda^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}.\end{aligned}$$

Since $\sigma_u(T_1+T_2) \leq 2\|T_1+T_2\|_{(1,\infty)} \log u$, it follows that

$$\delta' \leq 2\|T_1+T_2\|_{(1,\infty)} \int_\lambda^{2\lambda} \frac{du}{u} = \|T_1+T_2\|_{(1,\infty)} \int_1^2 \frac{du}{u} = \|T_1+T_2\|_{(1,\infty)} \log 2.$$

Combining this with (IV.3.26), (IV.3.27) and (IV.3.28) proves the lemma. □

Next, recall that $C_0[e,\infty)$ is a closed two-sided ideal of $C_b[e,\infty)$. Therefore, the quotient

$$\mathcal{Q} = C_b[e,\infty)/C_0[e,\infty).$$

is a (commutative) C^*-algebra with respect to the quotient norm,

$$\|[f]\|_{\mathcal{Q}} = \inf_{g \in C_0[e,\infty)} \|f+g\|_\infty \qquad \forall f \in C_b[e,\infty),$$

where $[f]$ denotes the class of f in $\mathcal{Q}$. Notice also that

$$\|[f]\|_{\mathcal{Q}} \leq \|f\| \qquad \forall f \in C_b[e,\infty). \tag{IV.3.29}$$

Notice also that $\mathcal{Q}$ is a commutative C^*-algebra.

We define a map $\tau : \mathcal{L}_+^{(1,\infty)} \to \mathcal{Q}$ by

$$\tau(T) = \text{class of } \lambda \to \tau_\lambda(T) \text{ in } \mathcal{Q} \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}.$$

Lemma IV.9. *The following hold.*

$$\tau(cT) = c\tau(T) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall c \geq 0. \tag{IV.3.30}$$

$$\tau(T_1+T_2) = \tau(T_1)+\tau(T_2) \qquad \forall T_j \in \mathcal{L}_+^{(1,\infty)}. \tag{IV.3.31}$$

$$\|\tau(T)\| \leq 2\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}, \tag{IV.3.32}$$

$$\tau(U^*TU) = \tau(T) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall U \in \mathcal{L}(\mathcal{H}),\ U \textit{ unitary.} \tag{IV.3.33}$$

Proof. Let T_1 and T_2 be in $\mathcal{L}_+^{(1,\infty)}$. It follows from Lemma IV.8 that

$$\tau_\lambda(T_1+T_2) = \tau_\lambda(T_1)+\tau_\lambda(T_2) \mod C_0[e,\infty),$$

and hence $\tau(T_1+T_2) = \tau(T_1)+\tau(T_2)$.

Let $T \in \mathcal{L}_+^{(1,\infty)}$. Then by (IV.3.25) and (IV.3.29) we have

$$\|\tau(T)\|_{\mathcal{Q}} \leq \sup_{\lambda \geq e} \tau_\lambda(T) \leq 2\|T\|_{(1,\infty)}.$$

In addition, for any $c \in [0,\infty)$, it follows from (IV.3.14) and (IV.3.23) that that $\tau_\lambda(cT) = c\tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(cT) = c\tau(T)$.

Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Then $U^*TU \in \mathcal{L}_+^{(1,\infty)}$ and by (III.1.6) $\mu_n(U^*TU) = \mu_n(T)$ for all $n \in \mathbb{N}_0$, and hence $\sigma_N(U^*TU) = \sigma_N(T)$ for all $N \in \mathbb{N}$. Combining this with (IV.3.11) and (IV.3.23) then shows that $\sigma_\lambda(U^*TU) = \sigma_\lambda(T)$ and $\tau_\lambda(U^*TU) = \tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(U^*TU) = \tau(T)$. The proof is complete. □

In the sequel we say that $x \in \mathcal{Q}$ is *positive* if $x = y^*y$ for some $y \in \mathcal{Q}$. Thus the condition (i) just says that a state on $\mathcal{Q}$ must take non-negative real values on positive elements of $\mathcal{Q}$.

It is not difficult to check that $x \in \mathbb{Q}$ if $x = [f]$ for some non-negative function $f \in C_b[0,\infty)$. In particular, for any $T \in \mathcal{L}_+^{(1,\infty)}$ the class $\tau(T)$ is a positive element of $\mathcal{Q}$.

We shall also write $x_1 \leq x_2$ to mean that $x_2 - x_1$ is positive. Notice that if $x \in \mathcal{Q}$ is selfadjoint, then $-\|x\|_{\mathcal{Q}}.1 \leq x \leq \|x\|_{\mathcal{Q}}.1$, for the functions $\|x\|_{\mathcal{Q}} \pm t$ are non-negative on $\operatorname{Sp} x \subset [0, \|x\|_{\mathcal{Q}}]$.

Definition IV.8. *A* state *on $\mathcal{Q}$ is a linear map $\omega : \mathcal{Q} \to \mathbb{C}$ such that*

(i) *ω is* positive, *i.e., $\omega(x) \geq 0$ if x is positive;*

(ii) ω *is* normalized*, i.e.,* $\omega(1) = 1$.

We denote by $\Omega(\mathcal{Q})$ *the set of states on* $\mathcal{Q}$

EXAMPLE IV.9. Any character $\chi : \mathcal{Q} \to \mathbb{C}$ is a state on $\mathcal{Q}$. Thus, there are plenty of states on $\mathcal{Q}$.

LEMMA IV.10. *Let* ω *be a state on* $\mathcal{Q}$. *Then*

(i) $\omega(x^*) = \overline{\omega(x)}$ *for all* $x \in \mathcal{Q}$.

(ii) ω *is a continuous linear form on* $\mathcal{Q}$. *In fact,*

$$|\omega(x)| \leq \|x\|_{\mathcal{Q}} \qquad \forall x \in \mathcal{Q}. \tag{IV.3.34}$$

PROOF. Let $x \in \mathcal{Q}$ be selfadjoint. As above-mentioned $\|x\|_{\mathcal{Q}}.1 \pm x$ are positive, and hence using the positivity of ω and the fact that $\omega(1) = 1$ we get

$$0 \leq \omega(\|x\|_{\mathcal{Q}} \pm x) = \|x\|_{\mathcal{Q}}\omega(1) \pm \omega(x) = \|x\|_{\mathcal{Q}} \pm \omega(x).$$

Thus,

$$|\omega(x)| \leq \|x\|_{\mathcal{Q}} \qquad \forall x \in \mathcal{Q},\ x \text{ selfadjoint}. \tag{IV.3.35}$$

Assume now that x is any element of $\mathcal{Q}$ and let us write $x = x_1 + ix_2$ with $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$. Then

$$\omega(x) = \omega(x_1) + i\omega(x_2).$$

As x_1 and x_2 are selfadjoint, both $\omega(x_1)$ and $\omega(x_2)$ are real numbers, and hence

$$\omega(x^*) = \omega(x_1 - ix_2) = \omega(x_1) - i\omega(x_2) = \overline{\omega(x_1) + i\omega(x_2)} = \overline{\omega(x)}.$$

In particular,

$$\Re(\omega(x)) = \omega(x_1) = \omega\left(\frac{x + x^*}{2}\right). \tag{IV.3.36}$$

It remains to show that $|\omega(x)| \leq \|x\|_{\mathcal{Q}}$. Since this inequality is obvious when $\omega(x) = 0$, we may assume $\omega(x) \neq 0$. Set $\alpha = \frac{\overline{\omega(x)}}{|\omega(x)|}$. Then $\omega(\alpha x) = \alpha\omega(x) = |\omega(x)|$, and hence using (IV.3.36) we get

$$|\omega(x)| = \Re(\omega(\alpha x)) = \omega\left(\frac{(\alpha x) + (\alpha x)^*}{2}\right).$$

Since $\frac{1}{2}\left((\alpha x) + (\alpha x)^*\right)$ is selfadjoint, using (IV.3.35) we obtain

$$|\omega(x)| \leq \left\|\frac{(\alpha x) + (\alpha x)^*}{2}\right\|_{\mathcal{Q}} \leq \frac{1}{2}(\|\alpha x\|_{\mathcal{Q}} + \|\overline{\alpha}x^*\|_{\mathcal{Q}}) = \|x\|_{\mathcal{Q}},$$

completing the proof.

□

The states on $\mathcal{Q}$ should be thought of as generalizations of the limit at ∞ of a function on $[e, \infty)$, very much in the same way Banach limits are generalizations of the limit of sequence. More precisely, we have

PROPOSITION IV.4.

(1) $\Omega(\mathcal{Q})$ *separates the points of* $\mathcal{Q}$.

(2) *Let* $f \in C_b[e, \infty)$. *Then*

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow \omega([f]) = L \quad \forall \omega \in \Omega(\mathcal{Q}).$$

Proof. Let $x \in \mathcal{Q} \setminus 0$. Denote by $\operatorname{Sp} Q$ the spectrum of $\mathcal{Q}$, i.e., the set of characters on $\mathcal{Q}$ endowed with the weak topology of $\mathcal{Q}^*$. Since $\mathcal{Q}$ is a commutative C^*-algebra, the injectivity of the Gel'fand transform $G_{\mathcal{Q}} : \mathcal{Q} \to C(\operatorname{Sp} \mathcal{Q})$, implies that there exists $\chi \in \operatorname{Sp} Q$ such that $\chi(x) \neq 0$. Since χ is a state, this shows that $\Omega(\mathcal{Q})$ separates the points of $\mathcal{Q}$.

Let $f \in C_b[e, \infty)$. Then

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow f = L \bmod C_0[e, \infty) \Longleftrightarrow [f] = L \text{ in } \mathcal{Q}.$$

Since $\Omega(\mathcal{Q})$ separates the point of $\mathcal{Q}$ and $\omega(L) = L\omega(1) = L$, it follows that

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow \omega([f]) = \omega(L) \ \forall \omega \in \Omega(\mathcal{Q}) \Longleftrightarrow \omega([f]) = L \ \forall \omega \in \Omega(\mathcal{Q}),$$

completing the proof. □

Let ω be a state on $\mathcal{Q}$. We define a functional $\operatorname{Tr}_\omega : \mathcal{L}_+^{(1,\infty)} \to [0, \infty)$ by letting

$$\operatorname{Tr}_\omega T := \omega(\tau(T)) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}.$$

Notice that as ω is linear and τ

We shall now extend Tr_ω into a linear form on $\mathcal{L}^{(1,\infty)}$. To this end we recall the following result, a proof of which can be found in Chapter III.

Lemma IV.11. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$. Then*

(i) *For any $T \in \mathcal{L}(\mathcal{H})$, we have*

$$T \in \mathcal{I} \Longrightarrow |T| \in \mathcal{I} \longrightarrow T^* \in \mathcal{I}. \tag{IV.3.37}$$

(ii) *Any $T \in \mathcal{I}$ can be written in the form,*

$$T = (T_1 - T_2) + i(T_3 - T_4) \qquad \textit{with } T_j \in \mathcal{I} \cap \mathcal{L}(\mathcal{H})_+. \tag{IV.3.38}$$

Proposition IV.5. *The functional Tr_ω uniquely extends to a linear form*

$$\operatorname{Tr}_\omega : \mathcal{L}^{(1,\infty)} \longrightarrow \mathbb{C}.$$

Proof. Thanks to (IV.3.31) and the linearity of ω we have

$$\operatorname{Tr}_\omega(T_1 + T_2) = \operatorname{Tr}_\omega T_1 + \operatorname{Tr}_\omega T_2 \qquad \forall T_j \in \mathcal{L}_+^{(1,\infty)}. \tag{IV.3.39}$$

Let $T \in \mathcal{L}^{(1,\infty)}$. Thanks to (IV.3.38) we can write $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$. Let $T = T_1' - T_2' + i(T_3' - T_4')$ be another such decomposition with $T_j' \in \mathcal{L}_+^{(1,\infty)}$. Observe that

$$\frac{1}{2}(T + T^*) = T_1 - T_2 = T_1' - T_2' \quad \text{and} \quad \frac{1}{2i}(T + T^*) = T_3 - T_4 = T_3' - T_4',$$

and hence $T_1 + T_2' = T_1' + T_2$ and $T_3 + T_4' = T_3' + T_4$. Therefore, using (IV.3.39) we get

$$\begin{aligned} \operatorname{Tr}_\omega(T_1) + \operatorname{Tr}_\omega(T_2') &= \operatorname{Tr}_\omega(T_1') + \operatorname{Tr}_\omega(T_2'), \\ \operatorname{Tr}_\omega(T_3) + \operatorname{Tr}_\omega(T_4') &= \operatorname{Tr}_\omega(T_3') + \operatorname{Tr}_\omega(T_4'). \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4)) \\ &\qquad\qquad = \operatorname{Tr}_\omega(T_1') - \operatorname{Tr}_\omega(T_2') + i(\operatorname{Tr}_\omega(T_3') - \operatorname{Tr}_\omega(T_4')). \end{aligned}$$

This shows that the value of the right-hand side above is the same for any decomposition $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$, and hence depends only on T. We then define

$$\operatorname{Tr}_\omega(T) := \operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4)), \tag{IV.3.40}$$

where the T_j's are any operators in $\mathcal{L}_+^{(1,\infty)}$ such that $T = T_1 - T_2 + i(T_3 - T_4)$.

Let $S \in \mathcal{L}^{(1,\infty)}$ and let us write $S = S_1 - S_2 + i(S_3 - S_4)$ with $S_j \in \mathcal{L}_+^{(1,\infty)}$. Then $S + T = (S_1 + T_1) - (S_2 + T_2) + i\left((S_3 + T_3) - (S_4 + T_4)\right)$. Observe that each operator $S_j + T_j$ is in $\mathcal{L}^{(1,\infty)}$ and is positive by Corollary II.1. Therefore, using (IV.3.40) and (IV.3.39) we get

(IV.3.41)
$$\begin{aligned}
&\operatorname{Tr}_\omega(S+T) = \operatorname{Tr}_\omega(S_1+T_1) - \operatorname{Tr}_\omega(S_2+T_2) + i(\operatorname{Tr}_\omega(S_3+T_3) - \operatorname{Tr}_\omega(S_4+T_4)) \\
&= \operatorname{Tr}_\omega(S_1) - \operatorname{Tr}_\omega(S_2) + i(\operatorname{Tr}_\omega(S_3) - \operatorname{Tr}_\omega(S_4)) + \operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4)) \\
&\qquad = \operatorname{Tr}_\omega S + \operatorname{Tr}_\omega T,
\end{aligned}$$

showing that Tr_ω is additive on $\mathcal{L}^{(1,\infty)}$.

Next, combining (IV.3.30) with the linearity of ω we get

(IV.3.42)
$$\operatorname{Tr}_\omega(\lambda T) = \lambda \operatorname{Tr}_\omega T \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall \lambda \geq 0.$$

Let $T \in \mathcal{L}^{(1,\infty)}$ and let us write $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$. Let λ be a non-negative real number. Then $\lambda T = \lambda T_1 - \lambda T_2 + i(\lambda T_3 - \lambda T_4)$. Since each operator λT_j is positive, using (IV.3.40) and (IV.3.42) we get

(IV.3.43)
$$\begin{aligned}
\operatorname{Tr}_\omega(\lambda T) &= \operatorname{Tr}_\omega(\lambda T_1) - \operatorname{Tr}_\omega(\lambda T_2) + i(\operatorname{Tr}_\omega(\lambda T_3) - \operatorname{Tr}_\omega(\lambda T_4)) \\
&= \lambda\left(\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4))\right) = \lambda \operatorname{Tr}_\omega(T).
\end{aligned}$$

Notice also that $-T = (T_2 - T_1) + i(T_4 - T_3)$ and $iT = (T_4 - T_3) + i(T_1 - T_2)$, and hence from (IV.3.40) we get

$$\begin{aligned}
\operatorname{Tr}_\omega(-T) &= \operatorname{Tr}_\omega(T_2) - \operatorname{Tr}_\omega(T_1) + i(\operatorname{Tr}_\omega(T_4) - \operatorname{Tr}_\omega(T_3)) = -\operatorname{Tr}_\omega(T), \\
\operatorname{Tr}_\omega(iT) &= \operatorname{Tr}_\omega(T_4) - \operatorname{Tr}_\omega(T_3) + i(\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2)) = i \operatorname{Tr}_\omega(T).
\end{aligned}$$

Let $\lambda \in \mathbb{C}$ and let us write $\lambda = \lambda_1 - \lambda_2 + i(\lambda_3 - \lambda_4)$ with $\lambda_j \geq 0$. Then combining the additivity of Tr_ω with (IV.3.43)–(IV.3) gives

$$\begin{aligned}
&\operatorname{Tr}_\omega(\lambda T) = \operatorname{Tr}_\omega\left(\lambda_1 T + (-\lambda_2 T) + (i\lambda_3 T) + (-i\lambda_4 T)\right) \\
&\qquad = \operatorname{Tr}_\omega(\lambda_1 T) + \operatorname{Tr}_\omega(-\lambda_2 T) + \operatorname{Tr}_\omega(i\lambda_3 T) + \operatorname{Tr}_\omega(-\lambda_4 T) \\
&\qquad\qquad = (\lambda_1 - \lambda_2 + i\lambda_3 - i\lambda_4) \operatorname{Tr}_\omega T = \lambda \operatorname{Tr}_\omega T.
\end{aligned}$$

Together with (IV.3.41) this shows that Tr_ω is a linear map.

Finally, since (IV.3.38) shows that $\mathcal{L}_+^{(1,\infty)}$ spans $\mathcal{L}^{(1,\infty)}$, any linear map that agrees with Tr_ω on $\mathcal{L}_+^{(1,\infty)}$ must agree with Tr_ω on all $\mathcal{L}^{(1,\infty)}$. Thus Tr_ω, as defined in (IV.3.40), is the unique linear extension to $\mathcal{L}^{(1,\infty)}$ of $\operatorname{Tr}_{\omega|\mathcal{L}_+^{(1,\infty)}} = \omega \circ \tau$. The proof is complete. □

PROPOSITION IV.6. *The following hold.*

(i) Tr_ω *is a continuous linear form on* $\mathcal{L}^{(1,\infty)}$. *In fact,*
$$|\operatorname{Tr}_\omega T| \leq 2\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

(ii) Tr_ω *is positive, i.e.,* $\operatorname{Tr}_\omega T \geq 0$ *for all* $T \in \mathcal{L}_+^{(1,\infty)}$.

(iii) $\operatorname{Tr}_\omega T^* = \overline{\operatorname{Tr}_\omega T}$ *for all* $T \in \mathcal{L}^{(1,\infty)}$.

(iii) Tr_ω *is a trace, that is,*
$$\operatorname{Tr}_\omega(TA) = \operatorname{Tr}_\omega(AT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A \in \mathcal{L}(\mathcal{H}).$$

PROOF. Let $T \in \mathcal{L}_+^{(1,\infty)}$. Then $\tau(T)$ is a positive element of $\mathcal{Q}$. As ω is a state, and hence is a positive linear form, it follows that $\operatorname{Tr}_\omega(T) = \omega(\tau(T)) \geq 0$. Thus Tr_ω is positive. In addition, combining (IV.3.32) and (IV.3.34) gives

(IV.3.44)
$$|\operatorname{Tr}_\omega(T)| = \omega(\tau(T)) \leq \|\tau(T)\|_{\mathcal{Q}} \leq 2\|T\|_{(1,\infty)}.$$

Let $T \in \mathcal{L}^{(1,\infty)}$ be selfadjoint. Using (IV.3.37) we see that $|T| \pm T$ is an element of $\mathcal{L}^{(1,\infty)}_{+}$. Thus,

$$0 \leq \operatorname{Tr}_\omega(|T| \pm T) = \operatorname{Tr}_\omega |T| \pm \operatorname{Tr}_\omega T.$$

Therefore, using (IV.3.44) we get

$$|\operatorname{Tr}_\omega T| \leq \operatorname{Tr}_\omega |T| \leq 2\| |T| \|_{(1,\infty)} = 2\|T\|_{(1,\infty)}.$$

Granted this, we then can argue as in the proof of Lemma IV.10 to show that, for all $T \in \mathcal{L}^{(1,\infty)}$, we have

$$\operatorname{Tr}_\omega T^* = \overline{\operatorname{Tr}_\omega T} \qquad \text{and} \qquad |\operatorname{Tr}_\omega T| \leq \|T\|_{(1,\infty)}.$$

Next, let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Using (IV.3.33) we see that, for any $T \in \mathcal{L}^{(1,\infty)}_{+}$,

$$\operatorname{Tr}_\omega(U^*TU) = \omega(\tau(U^*TU)) = \omega(\tau(T)) = \operatorname{Tr}_\omega(T).$$

Thus $T \rightarrow \operatorname{Tr}_\omega(U^*T)$ is a linear form on $\mathcal{L}^{(1,\infty)}$ that agrees with Tr_ω on $\mathcal{L}^{(1,\infty)}_{+}$, and so by Proposition IV.5 it agrees with Tr_ω on all $\mathcal{L}^{(1,\infty)}$. Thus,

$$\operatorname{Tr}_\omega(U^*TU) = \operatorname{Tr}_\omega(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

Upon changing T by UT this shows that

$$\operatorname{Tr}_\omega(TU) = \operatorname{Tr}_\omega(UT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

Since by Lemma III.5 the unitary operators span $\mathcal{L}(\mathcal{H})$, it follows that

$$\operatorname{Tr}_\omega(TA) = \operatorname{Tr}_\omega(AT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A \in \mathcal{L}(\mathcal{H}),$$

that is, Tr_ω is a trace. The proof is complete. □

Definition IV.10. *The functional* Tr_ω *from Proposition IV.5 is called the Dixmier trace associated to* ω.

In the sequel we let $\mathcal{H}'$ be a separable Hilbert space.

Lemma IV.12. *Let* Φ *be a continuous* $*$*-homomorphism from* $\mathcal{L}(\mathcal{H})$ *to* $\mathcal{L}(\mathcal{H}')$ *satisfying (IV.2.5). Then*

$$\operatorname{Tr}_{\omega,\mathcal{H}'}(\Phi(T)) = \operatorname{Tr}_{\omega,\mathcal{H}}(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Proof. We know by Lemma IV.3 that ϕ induces an isometric linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Moreover, as by assumption Φ is a homomorphism of C^*-algebras, it maps $\mathcal{L}(\mathcal{H})_+$ to $\mathcal{L}(\mathcal{H}')_+$. Thus Φ maps $\mathcal{L}^{(1,\infty)}(\mathcal{H})_+$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')_+$.

Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}')_+$. As shown in proof of Lemma IV.3, the property (IV.2.5) implies that $\sigma_u(\Phi(T)) = \sigma_u(T)$ for all $u \geq e$. Incidentally, $\tau_\lambda(\Phi(T)) = \tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(\Phi(T)) = \tau(T)$, which immediately implies that $\operatorname{Tr}_{\omega,\mathcal{H}'}(\Phi(T)) = \operatorname{Tr}_{\omega,\mathcal{H}}(T)$.

It follows from all this that $\operatorname{Tr}_{\omega,\mathcal{H}'} \circ \Phi$ is a well-defined linear map on $\mathcal{L}^{(1,\infty)}$ which agrees with $\operatorname{Tr}_{\omega,\mathcal{H}}$ on $\mathcal{L}^{(1,\infty)}(\mathcal{H})_+$, and so it follows from Proposition IV.5 that $\operatorname{Tr}_{\omega,\mathcal{H}'} \circ \Phi$ and $\operatorname{Tr}_{\omega,\mathcal{H}}$ agree on all $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, proving the lemma. □

Proposition IV.7. *Let* $S : \mathcal{H}' \rightarrow \mathcal{H}$ *be a continuous linear isomorphism from* $\mathcal{H}'$ *onto* $\mathcal{H}$. *Then*

$$\operatorname{Tr}_{\omega,\mathcal{H}'}(S^{-1}TS) = \operatorname{Tr}_{\omega,\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Proof. Recall that by Proposition IV.3 the conjugation by S gives rise to a continuous isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. In addition, set $|S| = (S^*S)^{\frac{1}{2}}$ and $U = S|S|^{-1}$. As shown in the proof of Proposition IV.3 $|S|$ is an invertible element of $\mathcal{L}(\mathcal{H}')$ and U is a unitary element of $\mathcal{L}(\mathcal{H}', \mathcal{H})$.

Notice that, as $\mathrm{Tr}_{\omega,\mathcal{H}'}$ is a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, for any $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}')$, we have

$$\mathrm{Tr}_\omega(|S|^{-1}T|S|) = \mathrm{Tr}_\omega(T|S|S^{-1}) = \mathrm{Tr}_\omega(T). \tag{IV.3.45}$$

Furthermore, as U is unitary, Remark III.2 tells us that the conjugation by U satisfies (IV.2.5), and hence by Lemma IV.12 we have

$$\mathrm{Tr}_{\omega,\mathcal{H}'}(U^*TU) = \mathrm{Tr}_{\omega,\mathcal{H}}(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}). \tag{IV.3.46}$$

Since $S = U|S|$, combining (IV.3.45) and (IV.3.46) we see that, for all $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$,

$$\mathrm{Tr}_{\omega,\mathcal{H}'}(S^{-1}TS) = \mathrm{Tr}_{\omega,\mathcal{H}'}\left(|S|^{-1}(U^*TU)|S|\right) = \mathrm{Tr}_{\omega,\mathcal{H}'}\, U^*T|U) = \mathrm{Tr}_{\omega,\mathcal{H}}(T),$$

proving the proposition. □

In particular, if we let $\mathcal{H}'$ be the hilbert space with same underlying vector space as $\mathcal{H}$ and equipped with an equivalent inner product, and we let S be the identity map, then we obtain

Corollary IV.2. *The Dixmier trace Tr_ω does not depend on the choice of the inner product of $\mathcal{H}$.*

Definition IV.11. *An operator $T \in \mathcal{L}^{(1,\infty)}$ is said to be measurable if the value of $\mathrm{Tr}_\omega T$ is independent of the choice of the state ω.*

We denote by $\mathcal{M}$ the subspace of $\mathcal{L}^{(1,\infty)}$ consisting of all measurable operators.

Definition IV.12. *The Dixmier trace of $T \in \mathcal{M}$, denoted $⨍ T$, is defined by*

$$⨍ T := \mathrm{Tr}_\omega T, \qquad \omega \text{ any state on } \mathcal{Q}.$$

We shall refer to the functional $⨍ : T \to ⨍ T$ as *the Dixmier trace* on $\mathcal{M}$.

Proposition IV.8. *The following hold.*

(1) *$\mathcal{M}$ is a closed subspace of $\mathcal{L}^{(1,\infty)}$ on which the Dixmier trace $⨍$ is a continuous linear form.*
(2) *Let $\mathcal{H}'$ be a (separable) Hilbert space and let Φ be a continuous $*$-homomorphism from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}')$ satisfying (IV.2.5). Then $\Phi(\mathcal{M}(\mathcal{H})) \subset \mathcal{M}(\mathcal{H}')$ and*
$$⨍_{\mathcal{H}'} \Phi(T) = ⨍_{\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$
(3) *Let $S : \mathcal{H} \to \mathcal{H}'$ be a continuous linear isomorphism from $\mathcal{H}$ onto a Hilbert space $\mathcal{H}'$. Then $\mathcal{M}(\mathcal{H}') = S^{-1}\mathcal{M}(\mathcal{H})S$ and*
$$⨍_{\mathcal{H}'} S^{-1}TS = ⨍_{\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$
(4) *$\mathcal{M}$ and $⨍$ don't depend on the choice of the inner product on $\mathcal{H}$.*

Proof. By definition

$$\mathcal{M} = \bigcap_{\omega,\omega' \in \Omega(\mathcal{Q})} \{T \in \mathcal{L}^{(1,\infty)};\ \mathrm{Tr}_\omega(T) = \mathrm{Tr}_{\omega'} T\}.$$

Since each Dixmier trace Tr_ω is a continuous linear form on $\mathcal{L}^{(1,\infty)}$ it follows that $\mathcal{M}$ is a closed subspace of $\mathcal{M}$. In addition, since $⨍$ agrees on $\mathcal{M}$ with any Dixmier trace, it follows from Proposition IV.6 that $⨍$ is a continuous linear form on $\mathcal{M}$. This proves the first part of the proposition. The other parts immediately follow from Lemma IV.12 and Proposition IV.7. □

Proposition IV.9. *The following hold.*

(1) *For any* $T \in \mathcal{L}_{+}^{(1,\infty)}$,

$$\left(T \in \mathcal{M} \text{ and } \fint T = L\right) \Longleftrightarrow \lim_{\lambda\to\infty} \tau_{\lambda}(T) = L.$$

(2) *For any* $T \in \mathcal{K}$ *positive,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n<N} \mu_n(T) = L \Longrightarrow \left(T \in \mathcal{M} \text{ and } \fint T = L\right).$$

Proof. Let $T \in \mathcal{L}_{+}^{(1,\infty)}$. Then it follows from Proposition IV.4 that

$$\lim_{\lambda\to\infty} \tau_{\lambda}(T) = L \Leftrightarrow \left(\omega(\tau(T)) = L \ \forall \omega \in \Omega(\mathcal{Q})\right) \Leftrightarrow \left(T \in \mathcal{M} \text{ and } \fint T = L\right).$$

Next, let T be a positive compact operator such that

$$\lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T) = L.$$

Then $\sigma_N(T) = \mathrm{O}(\log N)$, i.e., T is contained in $\mathcal{L}^{(1,\infty)}$.

Let $\lambda \in [e, \infty)$ and set $N = [\lambda]$. Then we have

$$\frac{\sigma_\lambda(T)}{\log \lambda} \leq \frac{\sigma_{N+1}(T)}{\log N} = \frac{\log(N+1)}{\log N} . \frac{\sigma_{N+1}(T)}{\log(N+1)},$$

$$\frac{\sigma_\lambda(T)}{\log \lambda} \geq \frac{\sigma_{N}(T)}{\log(N+1)} = \frac{\log N}{\log(N+1)} . \frac{\sigma_{N}(T)}{\log N}.$$

Since $\frac{\log(N+1)}{\log N}$ and $\frac{\log N}{\log(N+1)}$ both converge to 1 as $N \to \infty$, it follows that

$$\lim_{\lambda\to\infty} \frac{\sigma_u(T)}{\log u} \longrightarrow L \qquad \text{as } \lambda \to \infty.$$

If $L \neq 0$, then as $\lambda \to \infty$ we have $\frac{\sigma_\lambda(T)}{\log \lambda}\frac{1}{\lambda} \sim \frac{L}{\lambda}$, and hence

$$\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} \sim L \int_e^\lambda \frac{du}{u} \sim L \log \lambda.$$

If $L = 0$, then $\frac{\sigma_\lambda(T)}{\log \lambda}\frac{1}{\lambda} = \mathrm{o}(\frac{1}{\lambda})$, and hence

$$\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} = \mathrm{o}\left(\int_e^\lambda \frac{du}{u}\right) = \mathrm{o}(\log \lambda).$$

In both cases, we deduce that

$$\tau_\lambda(T) = \frac{1}{\log \lambda} \int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} \longrightarrow L \qquad \text{as } \lambda \to \infty.$$

It then follows from the first part of the proposition that T is measurable and $\fint T = L$. The proof is complete. □

Proposition IV.10. *The following types of operators are measurable and have a vanishing Dixmier trace:*

(i) *Any operator in* $\mathcal{L}_0^{(1,\infty)}$.
(ii) *Any trace-class operator.*
(iii) *Any infinitesimal operator of order* > 1.

Proof. Since $\mathcal{L}_0^{(1,\infty)}$ contains $\mathcal{L}^1$ and the infinitesimal operators of order > 1, (ii) and (iii) follows from (i). Therefore, we only have to prove that if $T \in \mathcal{L}_0^{(1,\infty)}$ then T is measurable and its Dixmier trace is zero.

Since $\mathcal{L}_0^{(1,\infty)}$ is a two-sided ideal (cf. Remark IV.1), Lemma IV.11 allows us to write T as $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_0^{(1,\infty)} \cap \mathcal{L}(\mathcal{H})_+$. Since

$\sigma_N(T_j) = \mathrm{o}(\log N)$, it follows from Proposition IV.9 that T_j is measurable and have a vanishing Dixmier trace. By linearity the same is true for T. The proof is complete. $\square$

Next, we shall make use of the following Tauberian theorem to obtain a measurability criterion.

LEMMA IV.13 (See [**GVF**, Lemmas 7.19–7.20]). *Let $(\lambda_n)_{n\geq 0}$ be a nonincreasing sequence of positive real numbers such that*

(i) $\sum_{n\geq 0} \lambda_n^s < \infty$ *for all* $s > 1$.

(ii) $\lim_{s\to 1^+}(s-1)\sum_{n\geq 0}\lambda_n^s = 1$.

Then

$$\lim_{N\to\infty} \frac{1}{\log N}\sum_{n<N}\lambda_n = 1.$$

PROPOSITION IV.11. *Let $T \in \mathcal{K}$ be positive and assume there exists $p > 0$ such that the following holds*

(i) T^s *is trace-class for all* $s > p$.

(ii) $\lim_{s\to p^+}(s-p)\operatorname{Trace} T^s = L \neq 0$.

Then T^p is measurable and

$$⨍ T^p = \frac{L}{p}.$$

PROOF. For $n \in \mathbb{N}_0$ set $\lambda_n = pL^{-1}\mu_n(T^p)$. If $s > 0$, then by Proposition III.5 we have $\mu_n(T^p) = \mu_n(T^{ps})$ for all $n \in N_0$, and hence

$$\sum_{n\geq 0}\lambda_n^s = \sum_{n\geq 0}(pL^{-1})^s\mu_n(T^p)^s = (pL^{-1})^s\sum_{n\geq 0}\mu_n(T^{ps}) = (pL^{-1})^s\operatorname{Trace} T^{ps}.$$

Therefore, we see that

- The series $\sum_{n\geq 0}\lambda_n^s$ is convergent for all $s > 1$.
- As $s \to 1^+$ we have

$$\sum_{n\geq 0}\lambda_n^s = (pL^{-1})^s\operatorname{Trace} T^{ps} = (pL^{-1})(sp-p)^{-1}L + \mathrm{o}(1) = (s-1) + \mathrm{o}(1).$$

We then can apply Lemma IV.13 to deduce that $\lim_{N\to\infty}\frac{1}{\log N}\sum_{n<N}\lambda_n = 1$. Since $\mu_n(T^p) = Lp^{-1}\lambda_n$ this gives

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n<N}\mu_n(T^p) = \frac{L}{p}. \tag{IV.3.47}$$

It then follows from Proposition IV.9 that T^p is measurable and $⨍ T^p = \frac{L}{p}$, proving the proposition. $\square$

EXAMPLE IV.13. Let $\Delta = -(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2)$ be the (positive) Laplacian on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. For $k \in \mathbb{Z}^n$ set $e_k = (2\pi)^{-\frac{n}{2}}e^{ik.x}$. Then $(e_k)_{k\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. Furthermore,

$$\Delta e_k = |k|^2 e_k \qquad \forall k \in \mathbb{Z}^n.$$

Thus $(|k|^2)_{k\in\mathbb{Z}^n}$ is the family of eigenvalues of Δ counted with multiplicity. We order it in a non-decreasing sequence $(\lambda_j)_{j\geq 0}$, i.e., λ_j is the $(j+1)$'th eigenvalue of Δ counted with multiplicity. (In fact, as 0 is an eigenvalue with multiplicity 1, if $j \geq 1$ then λ_j is the j'th nonzero eigenvalue counted with multiplicity.)

LEMMA IV.14 (Weyl Asymptotic). *As $j \to \infty$ we have*

$$\lambda_j \simeq \left(\frac{j}{c}\right)^{\frac{2}{n}}, \qquad c := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{IV.3.48}$$

PROOF. For $\lambda > 0$ define

$$N_1(\lambda) = \#\{j \in \mathbb{N}_0;\ \lambda_j < \lambda\} \quad \text{and} \quad N_2(\lambda) = \#\{j \in \mathbb{N}_0;\ \lambda_j < \lambda\}.$$

Observe that

$$N_1(\lambda) = \#\{k \in \mathbb{Z}^n;\ |k|^2 < \lambda\} = 2^n \#\{k \in \mathbb{N}_0^n;\ |k| < \sqrt{\lambda}\}, \tag{IV.3.49}$$

$$N_2(\lambda) = \#\{k \in \mathbb{Z}^n;\ |k|^2 \leq \lambda\} = 2^n \#\{k \in \mathbb{N}_0^n;\ |k| \leq \sqrt{\lambda}\}. \tag{IV.3.50}$$

In addition, for any $j \in \mathbb{N}$, we have

$$N_1(\lambda_j) \leq j \leq N_2(\lambda_j). \tag{IV.3.51}$$

For $r > 0$ we denote by $B(0,r)$ the (open) ball of radius r about the origin in $\mathbb{R}^n$ and we set $B^+(0,r) = B(0,r) \bigcap [0,\infty)^n$. For $k = (k_1,\ldots,k_n)$ in $\mathbb{N}_0^n$ we set $I_k := [k_1, k_1+1) \times \ldots \times [k_n, k_n+1)$. In addition, for any Borel set A we shall denote by $|A|$ its Lebesgue measure. For instance $|I_k| = 1$ and $|B(0,r)| = 2^{-n}cr^n$, with $c = |B(0,1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

If $x \in I_k$, then $|k| \leq |x| < |k| + \sqrt{n}$. Therefore, for any $\lambda > 0$, we have

$$B^+(0,\sqrt{\lambda}) \subset \bigcup_{|k|<\sqrt{\lambda}} I_k \qquad \text{and} \qquad \bigcup_{|k|\leq\sqrt{\lambda}} I_k \subset B^+(0, \sqrt{\lambda} + \sqrt{n}).$$

Thus,

$$|B^+(0,\sqrt{\lambda})| \leq \Big| \bigcup_{|k|<\sqrt{\lambda}} I_k \Big| = \#\{k \in \mathbb{N}_0^n;\ |k| < \sqrt{\lambda}\},$$

$$|B^+(0,\sqrt{\lambda}+\sqrt{n})| \geq \Big| \bigcup_{|k|\leq\sqrt{\lambda}} I_k \Big| = \#\{k \in \mathbb{N}_0^n;\ |k| \leq \sqrt{\lambda}\}.$$

Since $|B^+(0,r)| = 2^{-n}|B(0,r)| = 2^{-n}cr^n$, using (IV.3.49)–(IV.3.50) we deduce that

$$c\lambda^{\frac{n}{2}} \leq N_1(\lambda) \qquad \text{and} \qquad N_2(\lambda) \leq c\lambda^{\frac{n}{2}} \left(1 + \frac{\sqrt{n}}{\sqrt{\lambda}}\right)^n.$$

Combining this with (IV.3.51) shows that, for any $j \in \mathbb{N}$, we have

$$c\lambda_j^{\frac{n}{2}} \leq N_1(\lambda_j) \leq j \leq N_2(\lambda_j) \leq c\lambda_j^{\frac{n}{2}} \left(1 + \frac{\sqrt{n}}{\sqrt{\lambda_j}}\right)^n.$$

Thus,

$$\left(1 + \frac{\sqrt{n}}{\sqrt{\lambda_j}}\right)^{-\frac{1}{2}} \leq \lambda_j (cj^{-1})^{\frac{2}{n}} \leq 1. \tag{IV.3.52}$$

As the sequence $(\lambda)_{j\geq 0}$ is non-decreasing and unbounded it converges to ∞ and $(1 + \frac{\sqrt{n}}{\sqrt{\lambda_j}})^{-\frac{1}{2}}$ converges to 1 as $j \to \infty$. Combining this with (IV.3.52) shows that

$$\lambda_j (cj^{-1})^{\frac{2}{n}} \longrightarrow 1 \qquad \text{as } j \to \infty,$$

proving the claim. □

Next, the operator $\Delta^{-\frac{n}{2}}$ as the bounded operator on $L^2(\mathbb{T}^n)$ such that

$$\Delta^{-s} e_0 = 0 \qquad \text{and} \qquad \Delta^{-s} e_k = |k|^{-n} e_k \quad \forall k \in \mathbb{Z}^n \setminus 0.$$

Therefore, using Proposition II.5 (iv) we see that $\Delta^{-\frac{n}{2}}$ is a positive compact operator. Moreover, as the definition of $\Delta^{-\frac{n}{2}}$ implies that $((\lambda_j)^{-\frac{n}{2}})_{j\geq 1}$ is the non-increasing sequence of its eigenvalues with multiplicity, the min-max principle insures us that, for all $j \in \mathbb{N}_0$,

$$\mu_j(\Delta^{-\frac{n}{2}}) = (j+1)\text{'th eigenvalue of } \Delta^{-\frac{n}{2}} \text{ counted with multiplicity} = (\lambda_{j+1})^{-\frac{n}{2}}.$$

Combining this with (IV.3.48) shows that $\mu_j(\Delta^{-\frac{n}{2}}) \simeq \frac{c}{j}$ as $j \to \infty$. Thus,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \mu_j(\Delta^{-\frac{n}{2}}) = c.$$

We then may use Proposition IV.9 to deduce that $\Delta^{-\frac{n}{2}}$ is measurable and

$$⨍ \Delta^{-\frac{n}{2}} = c = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Example IV.14. Let us present an example of an operator in $\mathcal{L}^{(1,\infty)}$ which is not measurable. To this end, let σ be the function on $(1,\infty)$ defined by

$$\sigma(t) = \log t \left\{4 + \cos\left(\log(\log t)\right) - \sin\left(\log(\log t)\right)\right\}.$$

Notice that σ is a smooth function and

$$\sigma'(t) = \frac{2}{t}\left\{2 - \sin\left(\log(\log t)\right)\right\}$$

$$\sigma''(t) = \frac{-2}{t^2}\left\{2 - \sin\left(\log(\log t)\right) - (\log t)^{-1}\cos\left(\log(\log t)\right)\right\}.$$

Using (IV.14) we see that $\sigma(t)$ is increasing on $(1,\infty)$ and

$$|\sigma'(t)| \leq \frac{6}{t} \qquad \forall t \geq 1. \tag{IV.3.53}$$

Thus,

$$|\sigma(n+1) - \sigma(n)| \leq \frac{6}{n} \qquad \forall n \in \mathbb{N}. \tag{IV.3.54}$$

In addition, using (IV.14) we can check that $\sigma(t)$ is concave on (e,∞) and hence, for any integer $n \geq 3$, we have

$$\frac{1}{2}(\sigma(n+2) + \sigma(n)) \leq \sigma(n+1)$$

that is,

$$\sigma(n+2) - \sigma(n+1) \leq \sigma(n+1) - \sigma(n). \tag{IV.3.55}$$

Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$ be defined by

$$T\xi_n = \begin{cases} (\sigma(4) - \sigma(3))\xi_n & \text{for } n = 0,1,2, \\ (\sigma(n+1) - \sigma(n))\xi_n & \text{for } n \geq 3. \end{cases} \tag{IV.3.56}$$

The operator T is positive and using Proposition II.5-(iv) and (IV.3.54) we see that T is a compact operator. Furthermore, by (IV.3.55) the sequence $(\sigma(n+1) - \sigma(n))_{n\geq 3}$ is decreasing, and so using (IV.3.56) and the min-max principle we deduce that

(IV.3.57)

$$\mu_0(T) = \mu_1(T) = \mu_2(T) = \sigma(4) - \sigma(3) \quad \text{and} \quad \mu_n(T) = \sigma(n+1) - \sigma(n) \text{ for } n \geq 3.$$

Combining this with (IV.3.54) we see that that $\mu_n(T) = \mathrm{O}(\frac{1}{n})$, and hence T is an element of $\mathcal{L}^{(1,\infty)}$.

Let N be an integer ≥ 4. Then (IV.3.57) immediately implies that

$$\sigma_N(T) = \sum_{n<N} \mu_n(T) = \sigma(N) + a, \qquad a := 3\sigma(4) - 4\sigma(3).$$

Let $u \in [4,\infty)$ and set $N = [u]$ and $\alpha = u - N$. Then combining (IV.3.11) and (IV.14) we get

$$\sigma_u(T) = \alpha\sigma_{N+1}(T) + (1-\alpha)\sigma_N(T) = \alpha\sigma(N+1) + (1-\alpha)\sigma(N) + a,$$

and hence

$$|\sigma_u(T) - \sigma(u) - a| \leq \alpha|\sigma(N+1) - \sigma(u)| + (1-\alpha)|\sigma(N) - \sigma(u)|.$$

Thanks to (IV.3.53) we have

$$|\sigma(N+1) - \sigma(u)| \leq \frac{6}{u},$$

$$|\sigma(N) - \sigma(u)| \leq \frac{6}{N} \leq \frac{N+1}{N}\frac{6}{N+1} \leq \frac{5}{4}\frac{6}{u} = \frac{15}{2u},$$

where we have used the fact that $\frac{t+1}{t} \leq \frac{5}{4}$ for all $t \geq 4$. Thus,

$$|\sigma_u(T) - \sigma(u) - a| \leq \frac{15}{2u} \qquad \forall u \geq 4,$$

and hence

$$\sigma_u(T) = \sigma(u) + \mathrm{O}(1) \qquad \text{as } u \to \infty.$$

It follows from this that, as $\lambda \to \infty$, we have

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} + \mathrm{O}\left(\frac{1}{\log\lambda}\int_e^\lambda \frac{du}{u\log u}\right).$$

Since $\int_e^\lambda \frac{du}{u\log u} = \int_1^{\log\lambda}\frac{dv}{v} = \log\log\lambda = \mathrm{o}(\log\lambda)$, we see that

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} + \mathrm{o}(1). \tag{IV.3.58}$$

Next, we have

$$\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} = \int_1^{\log\lambda}\frac{\sigma(e^u)}{u}du = \int_1^{\log\lambda}(4+\cos(\log u) - \sin(\log u))\,du.$$

Observing that $\frac{d}{du}\left[u\,(4+\cos(\log u))\right] = (4+\cos(\log u) - \sin(\log u))$, we get

$$\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} = \log\lambda\left\{4+\cos\left(\log(\log\lambda)\right)\right\} - 4.$$

Combining this with (IV.3.58) we then obtain

$$\tau_\lambda(T) = 4 + \cos\left(\log(\log\lambda)\right) + \mathrm{o}(1) \qquad \text{as } \lambda \to \infty.$$

Therefore $\tau_\lambda(T)$ does not have a limit as $\lambda \to \infty$. It then follows from Proposition IV.9 that T is an element of $\mathcal{L}^{(1,\infty)}$ which is not measurable.

Bibliography

[Co] Connes, A.: *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[CM] Connes, A.; Moscovici, H.: *The local index formula in noncommutative geometry.* Geom. Funct. Anal. **5** (1995), no. 2, 174–243.

[Di] Dixmier, J.: *Existence de traces non normales.* C. R. Acad. Sci. Paris Sér. A-B **262** (1966) A1107–A1108.

[GVF] Gracia-Bondía, J.M.; Várilly, J.C.; Figueroa, H.: *Elements of noncommutative geometry.* Birkhäuser Boston, Boston, MA, 2001.

Chap. 5: Pseudodifferential Operators

1. Differential Operators:

$U \subset \mathbb{R}^n$ open.

Differential operator: lin. op. $P: C^\infty(U) \to C^\infty(U)$ of the form

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha, \quad a_\alpha(x) \in C^\infty(U)$$

where $D_{x_j} = \frac{1}{i}\partial_{x_j}$ and $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ $\alpha \in (\alpha_1, \dots, \alpha_n) \in (\mathbb{N}_0)^n$.

Fourier / Inverse Fourier transform

$$\hat{u}(\xi) = \int e^{-i\xi \cdot x} u(x)\, dx \qquad \check{v}(x) = \int e^{ix\cdot\xi} v(\xi)\, đ\xi, \quad đ\xi := (2\pi)^{-n} d\xi$$

For $u \in C_c^\infty(U)$:

$$(D_{x_j} u)^\wedge = \xi_j u \qquad D_{x_j} u = (\xi_j u)^\vee \qquad D_x^\alpha u = (\xi^\alpha u)^\vee$$

$$Pu = \sum a_\alpha(x) D_x^\alpha u = \sum a_\alpha(x) (\xi^\alpha u)^\wedge = \sum a_\alpha(x) \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi)\, đ\xi$$

$$\leadsto Pu(x) = \int e^{ix\cdot\xi} p(x,\xi) \hat{u}(\xi)\, đ\xi$$

where $p(x,\xi) := \sum a_\alpha(x) \xi^\alpha$ = symbol of P.

We also write: $P = p(x, D)$.

2. Classical Symbols:

Def.: $S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of functions $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$, that admit an asymptotic expan.

$$p(x,\xi) \sim \sum_{j \ge 0} p_{m-j}(x,\xi)$$

where:

- $p_{m-j}(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \setminus 0))$ and $p_{m-j}(x, \lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi)$ $\forall \lambda > 0$.
- the expan. $\sim$ means that, $\forall K \subset U$ compact $\forall N \in \mathbb{N}_0$ $\forall \alpha, \beta \in (\mathbb{N}_0)^n$, $\exists C_{KN\alpha\beta} > 0$ s.t.

$$\Big|\partial_x^\alpha \partial_\xi^\beta \Big(p - \sum_{j<N} p_{m-j}\Big)(x,\xi)\Big| \le C_{KN\alpha\beta} |\xi|^{\mathrm{Re}(m) - N - |\beta|} \quad \forall x \in K\ \forall \xi \in \mathbb{R}^n, |\xi| \ge 1.$$

Ex.: (1) If $m \in \mathbb{N}_0$, then $p(x,\xi) = \sum a_\alpha(x) \xi^\alpha \in S^m(U \times \mathbb{R}^n)$

(2) $(1 + |\xi|^2)^{m/2} \sim \sum_{j \ge 0} \binom{m/2}{j} |\xi|^{m-2j} \in S^m(U \times \mathbb{R}^n)$ $\forall m \in \mathbb{R}$.

Def. $S^{-\infty}(U \times \mathbb{R}^n)$ consists of $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ s.t., $\forall K \subset U$ compact $\forall N \in \mathbb{N}_0$ $\forall \alpha\beta \in \mathbb{N}_0^n$, $\exists C_{KN\alpha\beta} > 0$ s.t.

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \le C_{KN\alpha\beta} (1 + |\xi|)^{-N} \quad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

Rmk.: (1) $S^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{C}} S^m(U \times \mathbb{R}^n)$

(2) For two symbols $p \sim \sum p_{m-j}$ and $q \sim \sum q_{m'-j}$ we have

$p = q \mod S^{-\infty} \iff m = m'$ and $p_{m-j} = q_{m-j}$ $\forall j \in \mathbb{N}_0$.

3. <u>Pseudodifferential Operators:</u>

For $p(x,\xi) \in S^m(U\times\mathbb{R}^n)$ we denote by $p(x,D): C_c^\infty(U) \to C^\infty(U)$ the lin. op. defined by

$$p(x,D)u(x) = \int e^{ix\cdot\xi} p(x,\xi)\,\hat{u}(\xi)\, đ\xi \qquad \forall u \in C_c^\infty(U).$$

<u>Prop:</u> $p(x,D)$ is a continuous lin. op.

<u>Def</u> (A lin. op. $R: C_c^\infty(U) \to C^\infty(U)$ is called a <u>smoothing operator</u> when it admits a smooth kernel $k_R(x,y) \in C^\infty(U\times U)$,

$$Ru(x) = \int k_R(x,y)\,u(y)\,dy \qquad \forall u \in C_c^\infty(U)$$

The space of smoothing operators is denoted $\Psi^{-\infty}(U)$.

<u>Prop.</u> $p(x,D) \in \Psi^{-\infty}(U) \iff p(x,\xi) \in S^{-\infty}(U\times\mathbb{R}^n)$.

<u>Def.</u> (ΨDOs): $\Psi^m(U)$, $m\in\mathbb{C}$, consists of lin. op. $P: C_c^\infty(U) \to C^\infty(U)$ of the form

$$P = p(x,D) + R$$

where $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$ is also in $S^m(U\times\mathbb{R}^n)$ and R is smoothing op.

<u>Rmk:</u> (1) $p(x,\xi)$ is called the symbol of P, but it is not unique, only its class mod $S^{-\infty}(U\times\mathbb{R}^n)$ is unique.

(2) The leading coefficients are uniquely determined by P. The symbol $p_m(x,\xi)$ is called the <u>principal symbol</u> of P.

<u>Rmk:</u> We also can define ΨDOs acting on vector-valued functions, i.e., as ops $P: C_c^\infty(U,\mathbb{C}^r) \to C^\infty(U,\mathbb{C}^r)$ by using matrix-valued symbols $p(x,D)u(x) = \int e^{ix\cdot\xi} \underbrace{\underbrace{p(x,\xi)}_{M_r(\mathbb{C})}\,\underbrace{\hat{u}(\xi)}_{\mathbb{C}^r}}_{\mathbb{C}^r}$.

4. <u>Schwartz Kernels of ΨDOs:</u>

<u>Thm (Schwartz):</u> For a lin. op. $P: C_c^\infty(U) \to \mathcal{D}'(U)$ TFAE:

(i) P is continuous.

(ii) $\exists!$ kernel $k_P(x,y) \in \mathcal{D}'(U\times U)$ s.t.

$$\langle Pv(x), u(x)\rangle = \langle k_P(x,y), u(x)v(y)\rangle \quad \forall u,v \in C_c^\infty(U).$$

<u>Ex:</u> For $P = \sum a_\alpha D_x^\alpha \leadsto k_P(x,y) = \sum a_\alpha(x)(D_x^\alpha \delta_0)(x-y)$, δ_0 = Dirac function.

Let $p(x,\xi) \in S^m(U\times\mathbb{R}^n)$. If $\operatorname{Re}(m) < -n$, then we define

$$\check{p}_{\xi\to y}(x,y) := \int e^{iy\cdot\xi} p(x,\xi)\, đ\xi$$

$$\leadsto \check{p}_{\xi\to y}(x,x-y) = \int e^{i(x-y)\cdot\xi} p(x,\xi)\, đ\xi.$$

If $\operatorname{Re}(m) \geq -n$, then $\check{p}_{\xi\to y}(x,y)$ and $\check{p}_{\xi\to y}(x,x-y)$ make sense as elements of $C^\infty(U,\mathcal{S}'(\mathbb{R}^n)) \subset \mathcal{D}'(U\times\mathbb{R}^n)$

$$\langle \check{p}_{\xi\to y}(x,y), u(y)\rangle := \langle p(x,\xi), \check{u}(\xi)\rangle \qquad \forall u \in \mathcal{S}(\mathbb{R}^n)$$

$$\langle \check{p}_{\xi\to y}(x,x-y), u(y)\rangle := \langle \check{p}_{\xi\to y}(x,y), u(x-y)\rangle$$

<u>Prop.</u> Let $p(x,\xi) \in S^m(U\times\mathbb{R}^n)$. Then the kernel of $P = p(x,D)$ is

$$k_P(x,y) = \check{p}_{\xi\to y}(x,x-y).$$

Set $\Gamma = \{(x,x);\ x\in U\}\subset U\times U$.

<u>Prop.</u> Let $P\in\Psi^m(U)$ have kernel $k_P(x,y)$. Then:

(1) $k_P(x,y)$ is C^∞ on $U\times U\setminus\Gamma$.

(2) Assume $\mathrm{Re}(m)<-n$ and set $k=-([\mathrm{Re}(m)]+n+1)$. Then

$$k_P(x,y)\in C^k(U\times U).$$

<u>Prop.</u> Let $P\in\Psi^m(U)$, $m\in\mathbb{Z}$, $m\geq -n$. Denote by $k_P(x,y)$ its kernel and by $p(x,\xi)\sim\sum p_{m-j}(x,\xi)$ its symbol. Then, near $y=x$,

$$k_P(x,y)=\sum |x-y|^{-(m+n)+j}a_j(x,\theta)-c_P(x)\log|x-y|+O(1),$$

where we have set $\theta:=\frac{x-y}{|x-y|}$ and:

- $a_j(x,\theta)\in C^\infty(U\times S^{n-1})$ depends only on $p_{m-j}(x,\xi)$;
- $c_P(x)=(2\pi)^{-n}\int_{S^{n-1}}p_{-n}(x,\xi)\,dS(\xi)$.

$\quad$ surface measure on S^{n-1}.

<u>5. ΨDOs on a manifold:</u>

Let $\phi: V\to U$ be a C^∞-diffeomorphism, where $V\subset\mathbb{R}^n$ open. For a lin. op. $P: C_c^\infty(U)\to C^\infty(U)$ define the pull back operator $\phi^*P: C_c^\infty(V)\to C^\infty(V)$ by

$$\phi^*Pv(x):=P(v\circ\phi^{-1})(\phi(x))\qquad\forall v\in C_c^\infty(V).$$

<u>Prop.:</u> Let $P\in\Psi^m(U)$ have principal symbol $p_m(x,\xi)$. Then:

(1) $\phi^*P\in\Psi^m(V)$

(2) The principal symbol of ϕ^*P is given by

$$p_m^\phi(x,\xi):=p_m(\phi(x),(\phi'(x)^t)^{-1}\xi)\qquad\forall(x,\xi)\in V\times(\mathbb{R}^n\setminus 0).$$

Let M be a C^∞-manifold of dimension n. Let $P: C_c^\infty(M)\to C^\infty(M)$ be a lin. op..

- If $V\subset M$ is open, the restriction $P|_V: C_c^\infty(V)\to C^\infty(V)$ is defined by
$$P|_V v:=(Pv)|_V\quad\forall v\in C_c^\infty(V).$$
- If $\kappa: M\supset V\xrightarrow{\sim}U\subset\mathbb{R}^n$ is a local chart, then P_κ is the unique lin. op. $C_c^\infty(U)\to C^\infty(U)$ s.t.
$$Pv(x)=P_\kappa(v\circ\kappa^{-1})(\kappa(x))\quad\forall v\in C_c^\infty(V)$$

That is, $P_\kappa:=\kappa_*(P|_V)$.

In other words: P_κ $(\kappa^{-1})^*$ is the operator P in the local coordinates defined by κ.

<u>Ex:</u> $g=$ Riem. metric $\quad\Delta_g=d^*d: C^\infty(M)\to C^\infty(M)=$ Laplacian.

In loc. coord. $g=\sum g_{ij}(x)\,dx^i\otimes dx^j\quad g^{-1}=(g_{ij}(x))^{-1}=(g^{ij}(x))\quad g^{-1}$ Riem. metric on T^*M.

$$\Delta_g u=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j u),\ \text{where}\ \sqrt{g}:=\sqrt{\det(g)}$$

Def.: $\Psi^m(M)$, $m\in\mathbb{C}$, consists of continuous lin. op. $P: C_c^\infty(M)\to C^\infty(M)$ s.t., for all local charts $\kappa: V\xrightarrow{\sim} U\subset\mathbb{R}^n$,

$$P_\kappa \in \Psi^m(U).$$

Rmk: The symbol $p^\kappa(x,\xi)\sim\sum p^\kappa_{m-j}(x,\xi)$ of P_κ is called the symbol of P in the loc. coord. defined by κ.
It depends on κ in a non-trivial way, but ...
Let $\kappa_1: V_1\to U_1$ be another loc. chart and let $\phi=\kappa\circ\kappa_1^{-1}: \overbrace{\kappa_1(V\cap V_1)}^{W_1}\to\overbrace{\kappa(V\cap V_1)}^{W}$ be the transition map from κ_1 to κ. Then:

- $P_{\kappa_1}=\phi^* P_\kappa$.
- $p_m^{\kappa_1}(x,\xi)=p_m^\kappa(\phi(x),(\phi'(x)^t)^{-1}\xi)$

$x\in W_1$: $\phi'(x): T_x W_1\to T_x W$

$\phi'(x)^t: T_x^* W\to T_x W_1$

$(\phi'(x)^t)^{-1}: T_x^* W_1\to T_x W$.

Therefore:

Prop.: $\exists!$ function $\sigma_m(P)(x,\xi)\in C^\infty(T^*M\setminus 0)$ s.t.:

(i) $\forall$ loc. chart $\kappa: V\to U$,

$$\sigma_m(P)(\kappa(x),(\kappa'(x)^t)^{-1}\xi)=p_m^\kappa(x,\xi)\quad\forall(x,\xi)\in U\times\mathbb{R}^n\setminus 0\subset T^*U\setminus 0.$$

(ii) $\sigma_m(P)(x,\lambda\xi)=\lambda^m\sigma_m(P)(x,\xi)\quad\forall\lambda>0$.

Ex: g Riem. mtr. In loc. coord. $\Delta_g^u=\frac{-1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j)=-g^{ij}\partial_i\partial_j+$ lower order terms $=g^{ij}D_{x_i}D_{x_j}+\dots$

$$\leadsto \sigma_2(\Delta_g)(x,\xi)=\sigma_2(g^{ij}D_{x_i}D_{x_j})=g^{ij}(x)\xi_i\xi_j=(\|\xi\|_{g(x)})^2,\quad \|\xi\|_g=\text{norm defined by } g^{-1}(x) \text{ on } T_x^*M.$$

Rmk: We also can define ΨDOs acting on sections of vector bundle E over M, i.e., lin. op.
$$P: C_c^\infty(M,E)\to C^\infty(M,E). \quad\leftarrow\text{space of } C^\infty\text{-sections}$$
In that case the principal is defined as a section $\sigma_m(P)(x,\xi)\in C^\infty(T^*M\setminus 0,\operatorname{End}(\pi^*E))$

$\pi^*E=\coprod_{(x,\xi)}E_x$ — lift of E to T^*M.
↳ vector bundle over T^*M.

6. Properties of ΨDOs:

Assume M compact. Any ΨDO then is a cont. lin. $P: C^\infty(M)\to C^\infty(M)$.

Prop.: Let $P_1\in\Psi^{m_1}(M)$ and $P_2\in\Psi^{m_2}(M)$. Then:
$$P_1P_2\in\Psi^{m_1+m_2}(M)\ \text{ and }\ \sigma_{m_1+m_2}(P_1P_2)(x,\xi)=\sigma_{m_1}(P)(x,\xi)\,\sigma_{m_2}(P)(x,\xi).$$

Assume P is endowed with a positive density $\rho(x) \in C^\infty(M, |\Lambda|(M))$.
smooth one.

Define inner product on $L^2(M)$ by

$$\langle u, v \rangle_\rho := \int u(x)\, \overline{v(x)}\, d\rho(x)$$

A formal adjoint of a lin. op. $P: C^\infty(M) \to C^\infty(M)$ is a lin. op. $P^*: C^\infty(M) \to C^\infty(M)$ s.t.

$$(Pu, v)_\rho = \langle u, P^* v \rangle_\rho \quad \forall u, v \in C^\infty(M).$$

Prop. Let $P \in \Psi^m(M)$. Then:

(1) P^* exists and is contained in $\Psi^{\bar{m}}(M)$

(2) $\sigma_{\bar{m}}(P^*)(x,\xi) = \overline{\sigma_m(P)(x,\xi)}$.

Rk: P formally selfadjoint $\Rightarrow m \in \mathbb{R}$ and $\sigma_m(P)(x,\xi) \in \mathbb{R}$ $\forall (x,\xi) \in T^*M \setminus 0$.

Prop. Let $P \in \Psi^m(M)$ and set $a = \mathrm{Re}(m)$.

(1) P uniquely extends to a cont. lin. op.

$$P: C^{-\infty}(M) \to C^{-\infty}(M)$$

(2) For any $s \in \mathbb{R}$, the op. P uniquely extends to a cont. lin. op.

$$P: L^2_{s+a}(M) \to L^2_s(M).$$

Cor.: Let $P \in \Psi^m(M)$, $\mathrm{Re}(m) \le 0$. Then:

(1) P uniquely ext to a bounded op.

$$P: L^2(M) \to L^2(M)$$

(2) If $\mathrm{Re}(m) < 0$, then the above op. is compact.

Proof: $a = \mathrm{Re}(m) < 0$, $s = -a$

$$P: \underset{L^2_{a-a}(M)}{\overset{\|}{L^2_0(M)}} \to L^2_{-a}(M) \underset{-a>0}{\hookrightarrow} L^2(M)$$

compact inclu. of $-a>0$ by Rellich thm.

7. Ellipticity and Parametrices:

Def: $P \in \Psi^m(M)$ is said to be elliptic when $\sigma_m(P)(x,\xi) \neq 0$ $\forall (x,\xi) \in T^*M \setminus 0$.

Prop. TFAE

(1) P is elliptic.

(2) P admits a parametrix $Q \in \Psi^{-m}(M)$, i.e.,

$$PQ = QP = 1 \mod \Psi^{-\infty}(M)$$

Moreover, if (1) and (2) hold, then

$$\sigma_{-m}(Q)(x,\xi) = \sigma_m(P)(x,\xi)^{-1}.$$

Proof of (2) $\Rightarrow$ (1): $PQ = 1 \Rightarrow 1 = \sigma_0(1) = \sigma_0(PQ) = \sigma_m(P)\sigma_{-m}(Q)$.
$\Rightarrow \sigma_{-m}(Q) = \sigma_m(P)^{-1}$.

Prop. Assume that $P \in \Psi^m(M)$ is elliptic and set $a = \mathrm{Re}(m)$. For any $u \in C^{-\infty}(M)$,

$$Pu \in C^\infty(M) \iff u \in C^\infty(M)$$

$$Pu \in L^2_s(M) \iff u \in L^2_{s+a}(M)$$

$c(\sigma_m(P))$.

Cor: If P is elliptic, then $\ker P \subset C^\infty(M)$.

8. Spectral theory of elliptic ΨDOs:

Let $P: C^\infty(M) \to C^\infty(M)$ be an elliptic ΨDO of order m, $\mathrm{Re}(m) > 0$. If $a \notin \mathbb{R}$, we set $a \in \mathrm{Re}(m)$.

We can regard P as an unbounded operator on $L^2(M)$ w/ domain $L^2_a(M)$

By ellipticity $Pu \in L^2(M) \iff u \in L^2_a(M)$

Implies P is closed, so we can define the spectrum $\mathrm{Sp}\,P$.

Prop. There are only two possibility for $\mathrm{Sp}\,P$

(1) $\mathrm{Sp}\,P = \mathbb{C}$

(2) $\mathrm{Sp}\,P$ consists an ~~discrete~~ unbounded set of isolated eigenvalues. Moreover each eigensp $\ker(P-\lambda)$ is a fin. dim. sub of $C^\infty(M)$.

Lemma: If P is formally selfadjoint as ΨDO, then it is selfadjoint as an unbounded operator on $L^2(M)$.

Follows from closedness of P.

Prop. Assume P selfadjoint. Then:

(1) $\mathrm{Sp}\,P$ consists of isolated real eigenvalues w/ finite

(2) Each eigenspace $E_\lambda(P) := \ker(P-\lambda)$ is a fin. dim. subsp. of $C^\infty(M)$

(3) We have the orthogonal decomposition

$$L^2(M) = \bigoplus_{\lambda \in \mathrm{Sp}\,P} \ker(P-\lambda).$$

Prop. Suppose that P is selfadjoint and $\sigma_m(P)(x,\xi) > 0 \ \forall (x,\xi) \in T^*M \setminus 0$. Then:

(1) $\exists c \in \mathbb{R}$ s.t. $\mathrm{Sp}\,P \subset [-c, \infty)$, i.e., P is bounded from below.

(2) $\mathrm{Sp}\,P$ consists of a sequence of real eigenvals conv. to ∞.

Thm (Weyl Asymptotics): Suppose P is selfadjoint and $\sigma_m(P) > 0$. For $k \in \mathbb{N}_0$ define

$\lambda_k(P) = (k+1)$-th eigenvalue of P counted w/ multiplicity.

Then, as $k \to \infty$,

$$\lambda_k(P) \sim \left(\frac{k}{c(\sigma_m(P))}\right)^{m/n},$$

where $c(\sigma_m(P))$ depends only the principal symbol of P

Ex: $P = \Delta_g$ $\quad c(\sigma_2(\Delta_g)) = \dfrac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2}+1)} \mathrm{Vol}_g(M)$ (Volume of (M,g))

compact
$M = C^\infty$ mfld of dim. n
$E =$ vector bundle over M of rank r

1. Logarithmic singularity and the noncommutative residue:

Set $\Psi^{\mathbb{Z}}(M,E) = \bigcup_{m\in\mathbb{Z}} \Psi^m(M,E)$.

Let $P \in \Psi^{\mathbb{Z}}(M,E)$, $m\in\mathbb{Z}$, and let $k_P(x,\cdot)$ be the kernel of P. In any loc. coordinates and trivialization, we have near $y = x$ we have

$$k_P(x,y) = \sum |x-y|^{-(m+n)-j} a_j(x,\theta) - c_P(x)\log|x-y| + O(1)$$

where $\theta = \frac{x-y}{|x-y|}$ and $a_j(x,\theta) \in C^\infty(U\times S^{n-1})$ depends of a symbol $p_{m-j}(x,\xi)$, $j \geq 0$,

$$c_P(x) = (2\pi)^{-n}\int_{S^{n-1}} p_{-n}(x,\xi)\,dS(\xi)$$

A priori the $a_j(x,\theta)$ and $c_P(x)$ depend on a trivial way on the loc. coord and triv.

Let $\phi: V \longrightarrow U$ be a C^∞-diffeomorphism between open subsets of $\mathbb{R}^n$.
Let $P\in\Psi^{\mathbb{Z}}(U)$. Then $\phi^*P \in \Psi^{\mathbb{Z}}(V)$.

Lemma. We have

$$(**) \qquad c_{\phi^*P}(x) = |\phi'(x)|\, c_P(\phi(x)) \qquad \forall x\in V.$$

$(**)$ = transformation law for densities. Therefore, we can prove:

Prop: Let $P\in\Psi^{\mathbb{Z}}(M,E)$. Then the coefficient $c_P(x)$ of the log. singularity of $k_P(x,\cdot)$ in $(*)$ makes sense intrinsically on M as an $\mathrm{End}(E)$-valued density, i.e., as an elmt of $C^\infty(M, \mathrm{End}(E)\otimes|\Lambda|(M))$.

Suppose M compact. The noncommutative residue is the functional

$$\mathrm{Res}: \Psi^{\mathbb{Z}}(M,E) \longrightarrow \mathbb{C}$$

defined by

$$\mathrm{Res}(P) := \int_M \underbrace{\mathrm{tr}_{E_x} c_P(x)}_{\in\, C^\infty(M,|\Lambda|(M))}.$$

$$\boxed{\mathrm{Res}\,\Delta_g^{-n/2} = 2\frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2})}\mathrm{Vol}_g M}$$

$$\mathrm{Res}\,\Delta_g^{-n/2} = 2\frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2})}\int v_g(x)$$

Ex: g = Riemannian metric, $\Delta_g = d^*d$ = Laplacian. We saw that $\sigma_2(\Delta_g)(x,\xi) = (|\xi|_g)^2$

$\sigma_{-n}(\Delta_g^{-n/2})(x,\xi) = \sigma_2(\Delta_g)(x,\xi)^{-n/2} = (|\xi|_g)^{-n}$. Locally:

$$c_{\Delta_g}(x) = (2\pi)^{-n}\int_{S^{n-1}} (|\xi|_g)^{-n}\,dS(x) = (2\pi)^{-n}\underbrace{|S^{n-1}|}_{\frac{2\pi^{n/2}}{\Gamma(n/2)}=}\sqrt{g(x)} \rightsquigarrow \text{Globally } c_{\Delta_g^{-n/2}}(x) \overset{n}{=} (4\pi)^{-n/2}\,\Gamma\left(\tfrac{n}{2}+1\right)^{-1} v_g(x)$$

Proposition (Guillemin, Wodzicki):
(1) The NC residue vanishes on $\Psi^{-(n+1)}(M,E)$ and on differential operators.
(2) For any diffeomorphism $\phi: M' \to M$,
$$\operatorname{Res}_{M'} \phi^* P = \operatorname{Res} P \quad \forall P \in \Psi^{\mathbb{Z}}(M,E).$$
(3) The NC residue is a trace, i.e.,
$$\operatorname{Res} PQ = \operatorname{Res} QP \quad \forall P, Q \in \Psi^{\mathbb{Z}}(M,E).$$
If M is connected, then

Thm. (Wodzicki): Any trace on the algebra $\Psi^{\mathbb{Z}}(M,E)$ is a constant multiple of the NC residue.

2. Holomorphic families of ΨDOs and complex powers:

$U \subset \mathbb{R}^n$ open, $\Omega \subset \mathbb{C}^n$ open.

Def.: A family $(p(z))_{z\in\Omega} \subset S^*(U\times\mathbb{R}^n)$ is holomorphic when:
(i) The order $w(z)$ of $p(z)$ depends analytically on z.
(ii) $\forall (x,\xi) \in U\times\mathbb{R}^n$ the function $z \to p(z)(x,\xi)$ is holomorphic.
(iii) The bounds of the symbol expansion $p(z) \sim \sum p_{w(z)-j}(z)$ are uniform w.r.t. z on any compact set $K \subset \Omega$.

Def.: A family $(P(z)) \subset \Psi^*(M,E)$ is holomorphic when, in any loc. coord. and loc. triv., it can be written in the form
$$P(z) = p(z)(x,D) + R(z) \qquad \forall z \in \Omega,$$
where $(p(z))_{z\in\Omega}$ is a holomorphic family of symbols and $(R(z))_{z\in\Omega}$ is a hol. family of smoothing operators (i.e., $R(z)$ is given by a C^∞-kernel depending analytically on z).

Assume M compact. All the properties of ΨDOs can be extended to the setting of hol. families.

Prop (1) If $(P_1(z))_{z\in\Omega}$ and $(P_2(z))_{z\in\Omega}$ are holomorphic families of ΨDOs, then so is $(P_1(z)P_2(z))_{z\in\Omega}$.
(2) Let $P \in \Psi^{\mathbb{Z}}(M,E)$. Then there always exists a hol. family $(P(z))_{z\in\mathbb{C}}$ s.t. $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$.

Rk: A family $P(z)$ like in (2) is called a holomorphic gauging for P.

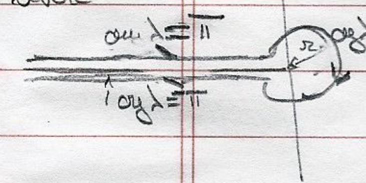
Let $P: C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic ΨDO of order m, $m>0$.

Assumption: Neither P nor its principal $\sigma_m(P)(x,\xi)$ have eigenvalues along the negative real-axis $(-\infty,0]$.

Ex: $P \geq 0$ and $\sigma_m(P)(x,\xi) > 0$.

<u>Lemma</u>: $\exists A>0$, $\lambda \to \infty$ along $(-\infty, 0]$,
$$\|(P-\lambda)^{-1}\| = O(|\lambda|^{-1}).$$

For $\mathrm{Re}\, s < 0$, we define a bounded operator by
$$P^s = \frac{-1}{2i\pi}\int_\Gamma \lambda^s (P-\lambda)^{-1}\, d\lambda.$$
where



$r>0$ st $\mathrm{Sp}\, P \cap \overline{D}(0,r) = \phi$.
$\lambda^s = e^{s\log(\lambda)}$.

<u>Lemma</u>: (1) $P^{s_1+s_2} = P^{s_1}P^{s_2}$
(2) $P^s|_{s=-k} = P^{-k} \quad \forall k \in \mathbb{N}$.

<u>Proof. (Seeley)</u> (1) P^s is a ΨDO of order ms
(2) $(P^s)_{\mathrm{Re}(s)<0}$ is a holomorphic family of ΨDOs.

We define P^s for any $s\in\mathbb{C}$ directly as the ΨDO
$$P^s = P^{s-k}\cdot P^k, \quad k \text{ any non-negative integer} > \mathrm{Re}(s).$$
The definition does not depend on k.

<u>Thm.</u> (Seeley) (1) P^s is a ΨDO of order ms s.t. $\sigma_{ms}(P^s)(x,\xi) = \sigma_m(P)(x,\xi)^s$.
(2) $P^{s_1+s_2} = P^{s_1}P^{s_2} \quad \forall s_j \in \mathbb{C}$
(3) $P^s|_{s=k} = P^k \quad \forall k\in\mathbb{Z}$. In particular, $P^s|_{s=0} = 1$.
(4) $(P^s)_{s\in\mathbb{C}}$ is a holomorphic family of ΨDOs.

<u>Cor</u>: (1) $P \geq 0$, $\sigma_m(P) > 0$, then $\sqrt{P} \in \Psi^{m/2}(M,E)$
(2) P elliptic selfadjoint, then $|P| = \sqrt{P^*P} \in \Psi^m(M,E)$

<u>Cor</u>: Let $A \in \Psi^a(M,E)$. Then $A(z) = AP^{z/m}$ is a hol. gauging for A.

<u>3. The NC residue as a residual trace</u> Assume M compact.

Define
$$\Psi^{\mathrm{int}}(M,E) = \bigcup_{\mathrm{Re}(m) < -n} \Psi^m(M,E).$$

Lemma: Let $P \in \Psi^{m}(M,E)$ have kernel $k_P(x,y)$.

(1) $k_P(x,y)$ is continuous

(2) The restriction $k_P(x,x)$ defines an $\operatorname{End} E_x$-valued density, i.e., an elt of $C^\infty(M, \operatorname{End}(E)\otimes|\Lambda|(M))$

(3) P is trace-class and

$$\operatorname{Trace}(P) = \int_M \operatorname{tr}_E k_P(x,x).$$

Rk: If in loc. coord $P = p(x,D)$, then

Proof

$$k_P(x,y) = \check{p}_{\xi\to y}(x, x-y) = \int e^{i(x-y)\cdot\xi} p(x,\xi)\, đ\xi \in C^0$$

$$k_P(x,x) = \int p(x,\xi)\, đ\xi \in C^\infty \quad \text{if } \operatorname{Re}(\operatorname{ord} P) < -n.$$

Set $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E) = \bigcup_{m\notin\mathbb{Z}} \Psi^m(M,E)$.

Proposition (Guillemin, Kontsevich-Vishik) (1) The map $\Psi^{m}(M,E) \ni P \longrightarrow k_P(x,x) \in C^\infty(M, \operatorname{End}(E)\otimes|\Lambda|(M))$ has a unique analytic extension to a map

$$\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E) \ni P \longrightarrow t_P(x) \in C^\infty(M, \operatorname{End}(E)\otimes|\Lambda|(M)).$$

(2) Let $P \in \Psi^{\mathbb{Z}}(M,E)$ and let $(P(z))_{z\in\mathbb{C}}$ be a hol. family of ΨDOs s.t. $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$.

Then $z \to t_{P(z)}(x)$ has at most a simple pole singularity near $z=0$ s.t.

$$\operatorname{Res}_{z=0} t_{P(z)}(x) = -c_P(x).$$

The canonical trace is the functional

$$\mathrm{TR}: \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E) \longrightarrow \mathbb{C}$$

defined by

$$\mathrm{TR}\, P := \int_M \operatorname{tr}_{E_x}(t_P(x)) \quad \forall P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E).$$

Thm (Guillemin, Wodzicki, Kontsevich-Vishik): (1) TR is the unique analytic ext. of the ordinary trace to $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,E)$

(2) $\mathrm{TR}[PQ] = \mathrm{TR}[QP] \quad \forall P,Q \in \Psi^{*}(M,E),\ \operatorname{ord}P + \operatorname{ord}Q \notin \mathbb{Z}$.

(3) Let $P \in \Psi^{\mathbb{Z}}(M,E)$ and let $(P(z))$ be a hol. gauging for P. Then $z \to \mathrm{TR}[P(z)]$ has at most a simple pole sing. near $z = 0$ s.t.

$$\operatorname{Res}_{z=0} \mathrm{TR}[P(z)] = -\operatorname{Res} P.$$

Rmk: Only (2) + (3) are enough to see that the NC residue is a trace.

Ex: $P = \Delta_g^{n/2}$

$\rightsquigarrow$ We saw that $\operatorname{Res} \Delta_g^{-n/2} = 2\frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2})} \operatorname{Vol}_g(M)$

$\rightsquigarrow \frac{1}{n} \operatorname{Res} \Delta_g^{-n/2} = \frac{(4\pi)^{-n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})} \operatorname{Vol}_g(M) = \frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2}+1)} \operatorname{Vol}_g(M)$

$\rightsquigarrow \lambda_k(\Delta_g) \sim \left(\frac{k}{c}\right)^{2/n}$ w/ $c = \frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2}+1)} \operatorname{Vol}_g(M)$

Application: Proof of the Weyl Asymptotics: $\mu_k(P) =$

$P \geq 0$, $\sigma_m(P) > 0$. For $\operatorname{Re}(s) > \frac{n}{m}$,

$$\sum \lambda_k(P)^{-s} = \operatorname{Trace} P^{-s} = \mathrm{TR}\left[P^{-\frac{n}{m}} P^{-(s-\frac{n}{m})}\right] = \frac{n \operatorname{Res}[P^{-n/m}]}{s-\frac{n}{m}} + O(1) \text{ near } s = \frac{n}{m}$$

$P(z) = P^{-n/m} \cdot P^{z/m}$ hol. gauging for $P^{-n/m}$ $\Longrightarrow$

Tauberian Thm. of Ikehara $\Longrightarrow \lambda_k(P) \sim \left[\frac{n}{\operatorname{Res} P^{-n/m}} \cdot k\right]^{m/n}$ as $k \to \infty$.

$\operatorname{Res} P^{-n/m}$ depends only symbol of order $-n$ of $P^{-n/m}$, but $P^{-n/m}$ has order $-n$, and then $\sigma_{-n}(P^{-n/m}) = \sigma_m(P)^{-n/m}$

$\rightsquigarrow \operatorname{Res} P^{-n/m}$ ________ on principal symbol of P.

Chap. 7: Quantized Calculus and Low Dimensional Volumes

1. The Dixmier Trace of a ΨDO:

M = compact manifold equipped w/ positive density $\rho>0$ as describing defining an inner product on $L^2(M)$

$$\langle u,v\rangle = \int_M \overline{u(x)}\,v(x)\,\rho(x)$$

Prop. Let $P\in\Psi^{-m}(M)$, $m>0$. Then

$$\mu_k(P) = O(k^{-m/n}) \quad \text{as } k\to\infty.$$

That is, P is an infinitesimal operator of order $\frac{m}{n}$.

$$\mu_k(P) = \mu_k(\Delta_g^{-m/2}\cdot\Delta_g^{m/2}P) \le_{2/n} \mu_k(\Delta_g^{-m/2})\,\|\Delta_g^{m/2}P\|$$

Proof: Let Δ_g be the Laplacian associated to a Riem. metric g on M. Weyl Asymptotic $\Rightarrow \lambda_k(\Delta_g)\sim\left(\frac{k}{c}\right)^{2/n} \Rightarrow \mu_k(P) = O(k^{-m/n})$

$$\Rightarrow \mu_k(\Delta_g^{-m/2}) = \lambda_k(\Delta_g)^{-m/2} = O(k^{-m/n})$$

$P = \Delta_g^{-m/2}\underbrace{\Delta_g^{m/2}P}_{\text{order } 2\cdot\frac{m}{2}-m=0}$ $\Rightarrow \Delta_g^{m/2}P$ bounded.

Thm. (Connes): Let $P\in\Psi^{-n}(M)$. Then P is measurable and

$$⨍P = \frac{1}{n}\operatorname{Res}(P).$$

Proof: Let $P\in\Psi^1(M)$ be a positive selfadjoint ΨDO of order 1 s.t. $\sigma_1(P)>0$ (so that P is elliptic).

Weyl asymptotic:

$$\lambda_k(P)\sim\left[\frac{n}{\operatorname{Res}P^{-n}}\cdot k\right]^{1/n}.$$

Therefore,

$$\mu_k(P^{-n}) = \lambda_k(P)^{-n}\sim\frac{1}{n}\operatorname{Res}[P^{-n}]\cdot\frac{1}{k}$$

Thus,

$$\sum_{k<N}\mu_k(P^{-n})\sim\frac{1}{n}\operatorname{Res}[P^{-n}]\cdot\log N \Rightarrow \operatorname{Tr}_\omega[P^{-n}] = \frac{1}{n}\operatorname{Res}[P^{-n}]$$

This proves Connes Thm. for P^{-n}.

In general we can write any $P\in\Psi^{-n}(M)$ in the form

$$P = (P_1)^{-n}-(P_2)^{-n}+i\left((P_3)^{-n}-(P_4)^{-n}\right)+Q$$

where: - $P_1,\dots,P_4$ are positive self. 1st order ΨDOs s.t. $\sigma_1(P_j)>0$.

- $Q\in\Psi^{-(n+1)}(M)$.

$\operatorname{Res}Q = 0$ + $P\in\Psi^{-(n+1)}(M) \Rightarrow Q$ inf. of order $\frac{n+1}{n}>1 \Rightarrow \operatorname{Tr}_\omega Q = 0$.

$$\Rightarrow \operatorname{Tr}_\omega(P) = \sum\operatorname{Tr}_\omega(P_j) = \sum\frac{1}{n}\operatorname{Res}(P_j) = \frac{1}{n}\operatorname{Res}P.$$

Important Consequence: We can extend the definition of the Dixmier trace to all ΨDOs of any order by letting

$$⨍P := \frac{1}{n}\operatorname{Res}P \quad \forall P\in\Psi^{\mathbb{Z}}(M).$$

That is, we can integrate any ΨDO even if it is not an inf. op. of order ≥ 1 !!!

2. Lower Dimensional Volumes in Riemannian Geometry (RP'08)

Let (M^n, g) be a compact Riem. mfld.

Volume of (M,g):
$$\mathrm{Vol}_g(M) := \int_M \underbrace{v_g(x)}_{\text{Riemannian density}}$$

Q: Can we define volume of dimension k for $k=1,2,\ldots,n$? E.g. can we define the area and length of a manifold in any dimension?

A: Yes, by using NCG.

2 cases: • dim M even/odd

Case I: dim M even

M spin $\Rightarrow$ we can define the Dirac operator $D\!\!\!/ : C^\infty(M, \$) \longrightarrow C^\infty(M,\$)$

Prop.
$$⨍ f D\!\!\!/^{-n} = \frac{(2\pi)^{-n}}{\Gamma(\frac{n}{2}+1)} \int_M f(x)\, v_g(x) \qquad \forall f \in C^\infty(M)$$

Proof: Lichnerowicz Formula $\Rightarrow D\!\!\!/^2 = \underbrace{(\nabla^{\$})^*\nabla^{\$}}_{\text{Bochner Laplacian on spinors}} + \frac{1}{4}\kappa \rightsquigarrow \sigma_2(D\!\!\!/^2) = \sigma_2((\nabla^{\$})^*\nabla^{\$}) = \underbrace{(|\xi|_g)^2}_{g^{ij}(x)\xi_i\xi_j \text{ metric on } T^*M} \cdot \mathrm{id}_{\$_x}$ $\quad = \sigma_2(\Delta_g)\ldots$

n even $\Rightarrow \sigma_{-n}(D\!\!\!/^{-n}) = \sigma_{-n}((D\!\!\!/^2)^{-n/2}) = \sigma_{-n}(D\!\!\!/^2)^{-n/2} = (|\xi|_g)^{-n}$

$\rightsquigarrow \sigma_{-n}(f D\!\!\!/^{-n}) = f(x)|\xi|_g^{-n}$

In local coord.:
$$c_{fD\!\!\!/^{-n}}(x) = (2\pi)^{-n} f(x) \int_{S^{n-1}} (|\xi|_g)^{-n}\, dS(\xi) = (2\pi)^{-n}\sqrt{g(x)}\,.\,|S^{n-1}| \quad \left(|S^{n-1}| = 2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} = n\,.\,\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right)$$

As a density
$$c_{fD\!\!\!/^{-n}}(x) = f(x)(4\pi)^{-n/2} n\,.\,\Gamma(\tfrac{n}{2}+1)^{-1} v_g(x)$$

$$⨍ f D\!\!\!/^{-n} = \frac{1}{n}\mathrm{Res}[f D\!\!\!/^{-n}] = \frac{1}{n}\int \mathrm{tr}_{\$_x}[c_{fD\!\!\!/^{-n}}(x)] = \frac{1}{n}\int_M n\,.\,\frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2}+1)}\,.\,\underbrace{\mathrm{rk}\,\$_x}_{2^{n/2}}\, f(x) v_g(x)$$
$$= \frac{(2\pi)^{-n/2}}{\Gamma(\frac{n}{2}+1)}\int_M f(x)\, v_g(x).$$

Consequence: The operator $(2\pi)^{n/2}\Gamma(\frac{n}{2}+1)D\!\!\!/^{-n}$ allows us to recover the volume form.

Def (Length Element): The length element is the operator
$$ds = \left[(2\pi)^{n/2}\Gamma(\tfrac{n}{2}+1) D\!\!\!/^{-n}\right]^{1/n} = c_n D\!\!\!/^{-1}, \quad c_n := \sqrt{2\pi}\ \Gamma(\tfrac{n}{2}+1)^{1/n}.$$

Rmk: In QFT $D\!\!\!/^{-1}$ = free propagator for Fermions $\rightsquigarrow$ dimension of a length.

Def: For $k=1,\ldots,n$, the k-th dimensional volume of (M,g) is
$$\mathrm{Vol}_g^{(k)} M := ⨍ ds^k.$$

Prop. (RP) (i) If k is odd, then
$$(*') \qquad \mathrm{Vol}_g^{(k)} M = 0.$$

Th(2) If k is even, then

$$(*')\quad \mathrm{Vol}_g^{(k)} M = \nu(n,k)\int_M \alpha_{\frac{n-k}{2}}(x)\, \nu_g(x), \qquad \nu(n,k) = \frac{k}{n}(2\pi)^{\frac{k-n}{2}} \frac{\Gamma(\frac{n}{2}+1)^{k/n}}{\Gamma(\frac{k}{2}+1)}$$

where $\alpha_{\frac{n-k}{2}}(x)$ is a universal polynomial in the covariant derivatives of the curvatures. E.g.,

$$\alpha_0(x) = 1, \quad \alpha_1(x) = -\frac{\kappa(x)}{12}, \quad \alpha_2(x) = \frac{-1}{1{,}440}\left(-5\kappa^2 + 8|Ric|_g^2 + 7|R|_g^2\right)$$

Here R_{ijkl} = curvature tensor $|R|_g^2$ = norm square w.r.t g

$Ric_{jk} = g^{il} R_{ijkl}$ = Ricci tensor $|Ric|_g^2$ = ———— g

$\kappa = g^{jk} Ric_{jk}$ = scalar curvature.

Observation: The right-hand sides of $(*)$ and $(*')$ depend only on the metric $\Rightarrow$ they do not depend on the spin str.

$\leadsto$ use these equations to define lower dimensional volumes when M is not spin.

Def. If M is not spin, then for $k = 1, \ldots, n$ we define

$$\mathrm{Vol}_g^{(k)} M = \begin{cases} \nu(n,k) \int_M \alpha_{\frac{n-k}{2}}(x)\, \nu_g(x) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Rmk: The above definition is purely differential-geometric; it does not involve any NCG or spin geometry.

Ex: (1) $\mathrm{Vol}_g^{(n)} M = \nu(n,n)\int_M \alpha_0(x)\, \nu_g(x) = \int_M \nu_g(x) = \mathrm{Vol}_g M.$

(2) The area of (M,g) is

$$\mathrm{Area}_g M := \mathrm{Vol}_g^{(2)} M.$$

In dim. 4:

$$\mathrm{Area}_g M^4 = \frac{-1}{24\pi\sqrt{2}} \int_M \kappa(x)\, \nu_g(x) = \text{Einstein-Hilbert action of gravity.}$$

In dim. 6:

$$\mathrm{Area}_g M^6 = \frac{-1}{34{,}560}\int_M \left(-5\kappa^2 + 8|Ric|_g^2 + 7|R|_g^2\right) \nu_g(x)$$

Case 2: dim M odd.

Prop: If M is spin then $\quad \displaystyle ⨍ f |D|^{-n} = \frac{(2\pi)^{-n/2}}{\sqrt{2}\,\Gamma(\frac{n}{2}+1)} \int_M f(x)\, \nu_g(x) \quad \forall f \in C^\infty(M)$

$\leadsto ds = c_n' |D|^{-1}, \quad c_n' := \sqrt[2n]{2}\cdot c_n = 2^{1/2n}(2\pi)^{-1/2}\sqrt[n]{\Gamma(\frac{n}{2}+1)}$

Same reasoning as in the even dim. case leads us to $\quad \mathrm{Vol}_g^{(k)}(M) = ⨍ ds^k.$

Def: For $k = 1, \ldots, n$ the k-th dim. volume is defined by

$$\mathrm{Vol}_g^{(k)}(M) = \begin{cases} \nu'(n,k) \int_M \alpha_{\frac{n-k}{2}}(x)\, \nu_g(x) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

where $\nu'(n,k) := 2^{\frac{k-n}{2n}}\, \nu(n,k).$

Ex: Define the length of (M,g) to be

$$\mathrm{Length}_g(M) := \mathrm{Vol}_g^{(1)}(M)$$

- In dim. 3:

$$\mathrm{Length}_g(M^3) = -\frac{\sqrt[3]{6}}{72\pi\sqrt[3]{\pi}}\int_M \kappa(x) - \tau_g(x)$$

- In dim. 5:

$$\mathrm{Length}_g(M^5) = -\frac{\pi\sqrt[5]{60}}{28{,}800\sqrt[9]{\pi}}\int_M \alpha_2(x).$$